Gaussian Distributions and the Fokker-Planck Equation

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Abstract: This paper elucidates the relationship between the parameters of the Gaussian distribution with the coefficients of the Fokker-Planck equation. It is found that generally, the variance of Gaussian distributions is related to the diffusion coefficient of the Fokker-Planck equation, while the drift coefficient of the Fokker-Planck equation is connected to the deviation from the mean of the Gaussian function. When only the standard deviation is time-dependent, the Gaussian distribution is a solution of a drift-free Fokker-Planck equation. When only the mean is time-dependent, the Gaussian distribution is a solution of a diffusion-free Fokker-Planck equation.

Key Words: Gaussian distributions; Fokker-Planck equation

1. INTRODUCTION

The Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of a stochastic variable. Previous works by the author on the dynamics of Philippine stock markets have shown that the probability density functions of stock prices or log returns obey the Fokker-Planck equation of various types. The probability density of PLDT stock price for example is Gaussian with time-depedent standard deviation which was shown to obey a drift-free Fokker-Planck equation (Kaw and Roleda 2014). The probability density of the log return of the San Miguel Corporation stock price on the other hand is log-logistic with a time-dependent scale parameter. This was shown to obey a diffusion-free Fokker Planck equation (Roleda and Ultra 2015). This paper is the first of a series that the author hopes to pursue to shed some light on the relationships between the parameters of probability density functions and the coefficients of the Fokker-Planck equation. By virtue of the central limit theorem, many distributions tend to converge to the normal distribution. It is therefore but apt that this program begins with the Gaussian distribution.

2. PRELIMINARIES

The Fokker-Planck equation is given by

$$\frac{\partial f}{\partial t} = -\frac{\partial \beta f}{\partial x} + \frac{\partial^2 Df}{\partial x^2} \qquad \text{(Eq. 1)}$$

where

- f : the probability density function
- x : the stochastic variable
- β : drift coefficient
- D : diffusion coefficient

The Gaussian (normal) distribution on the hand is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \qquad \text{(Eq. 2)}$$

where

- μ : mean
- σ : standard deviation

3. ANALYSIS

The conditions by which the Gaussian distribution (eqn. 2) is a solution of the Fokker-Planck equation (eqn. 1) are expounded in this section, taking into consideration the various functional forms of the distribution parameters μ and σ.

Case A. constant and homogeneous μ and σ

Since

$$\frac{\partial f}{\partial t} = 0$$

and

$$\frac{\partial f}{\partial x} = -\frac{(x-\mu)}{\sigma^2} f \qquad \text{(eqn. 3)}$$

The Fokker-Planck equation may be satistified since eqn. 3 can be recast as

$$\frac{\partial \sigma^2 f}{\partial x} = (\mu - x) f$$

Thus,

$$\frac{\partial^2 (\sigma^2 f)}{\partial x^2} = \frac{\partial (\mu - x) f}{\partial x}$$

implying that

$$D = \sigma^2 \qquad \text{(eqn. 4a)}$$

$$\beta = (\mu - x) \qquad \text{(eqn. 4b)}$$

Case B. homogeneous μ and σ and constant μ

For σ(*t*),

$$\frac{\partial f}{\partial t} = -\frac{\dot{\sigma}}{\sigma} f + \frac{(x-\mu)^2 \dot{\sigma}}{\sigma^3} f$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{f}{\sigma^2} + \frac{(x-\mu)^2}{\sigma^4} f \qquad \text{(eqn. 5)}$$

where

$$\dot{\sigma} \equiv \frac{\partial \sigma}{\partial t}$$

Thus,

$$\frac{\partial f}{\partial t} = \dot{\sigma}\sigma \frac{\partial^2 f}{\partial x^2}$$

implying that

$$D = \sigma \dot{\sigma} \qquad \text{(eqn. 6a)}$$

$$\beta = 0 \qquad \text{(eqn. 6b)}$$

Case C. homogeneous μ and σ and constant σ

For μ(*t*),

$$\frac{\partial f}{\partial t} = \frac{(x-\mu)\dot{\mu}}{\sigma^2} f$$

$$\frac{\partial f}{\partial x} = -\frac{(x-\mu)}{\sigma^2} f \qquad \text{(eqn. 7)}$$

Thus,

$$D = 0 \qquad \text{(eqn. 8a)}$$

$$\beta = \dot{\mu} \qquad \text{(eqn. 8b)}$$

Case D. homogeneous μ and σ

For μ(*t*) and σ(*t*),

$$\frac{\partial f}{\partial t} = -\frac{\dot{\sigma}}{\sigma} f + \frac{(x-\mu)^2 \dot{\sigma}}{\sigma^3} f + \frac{(x-\mu)\dot{\mu}}{\sigma^2} f$$

Since eqns. 5 and 7 still hold,

$$D = \sigma \dot{\sigma} \qquad \text{(eqn. 9a)}$$

$$\beta = \dot{\mu} \qquad \text{(eqn. 9b)}$$

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Case E. constant μ and σ and homogeneous σ

For μ(x),

$$\frac{\partial f}{\partial t} = 0$$

and

$$\frac{\partial f}{\partial x} = -\frac{(x-\mu)}{\sigma^2}(1-\mu')f$$

where

$$\mu' \equiv \frac{\partial \mu}{\partial x}$$

The Fokker-Planck equation may be satisfied if

$$D = \sigma^2 \qquad \text{(eqn. 10a)}$$

$$\beta = (\mu - x)(1-\mu') \qquad \text{(eqn. 10b)}$$

Case F. constant μ and σ and homogeneous μ

For σ(x),

$$\frac{\partial f}{\partial t} = 0$$

Noting that

$$\sigma^2 f = \frac{\sigma}{\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

so that

$$\frac{\partial \sigma^2 f}{\partial x} = \sigma\sigma' f + \frac{(x-\mu)^2\sigma^2\sigma'}{\sigma^3} f - (x-\mu)f$$

This may be recast as

$$\frac{\partial \sigma^2 f}{\partial x} = \beta f$$

so that

$$\frac{\partial^2 \sigma^2 f}{\partial x^2} = \frac{\partial \beta f}{\partial x}$$

Hence, the diffusion and drift coefficients are

$$D = \sigma^2 \qquad \text{(eqn. 11a)}$$

$$\beta = \sigma\sigma' - (x-\mu) + (x-\mu)^2\frac{\sigma'}{\sigma} \qquad \text{(eqn. 11b)}$$

Case G. constant but non-homogeneous μ and σ

For σ(x) and μ(x),

$$\frac{\partial f}{\partial t} = 0$$

while

$$\frac{\partial \sigma^2 f}{\partial x} = \sigma\sigma' f + \frac{(x-\mu)^2\sigma'}{\sigma} f - (x-\mu)(1-\mu')f$$

Hence,

$$D = \sigma^2 \qquad \text{(eqn. 12a)}$$

$$\beta = \sigma\sigma' - (x-\mu)(1-\mu') + (x-\mu)^2\frac{\sigma'}{\sigma} \qquad \text{(eqn. 12b)}$$

4. CONCLUSIONS

Gaussian distribution is a solution to the Fokker-Planck equation when its parameters are independent of time and the stochastic variable x, when its parameters are dependent on time alone, or when its parameters are dependent on the stochastic variable alone. For time-independent distributions, the diffusion coefficient is the variance of the distribution. The drift coefficient on the other hand is a function of deviation from the mean $(x-\mu)$. Although the distribution is strictly speaking no longer Gaussian if the parameters are x-dependent, it is nevertheless worth noting how the drift and diffusion coefficients may be identified when the probability density function is cast in a Gaussian-like form.

When the variance alone is time-dependent, the Gaussian distribution is a solution of a drift-free Fokker-Planck equation, and the diffusion coefficient is $\sigma\dot{\sigma}$. When the mean alone is time-dependent, the Gaussian distribution is a solution of a diffusion-free Fokker-Planck equation, and the drift coefficient is $\dot{\mu}$. When both variance and mean are time-dependent, the Gaussian distribution is also a solution of the Fokker-Planck equation. The drift coefficient remains $\dot{\mu}$ and the diffusion coefficeint remains $\sigma\dot{\sigma}$.

5. REFERENCES

Kaw, R.C. and Roleda, R.C. (2014) Dynamics of PLDT Stock Price. Proceedings of the DLSU Research Congress 2014.

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