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On The Square of an Oriented Graph Conjecture

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Abstract: In 1993, Paul Seymour posed the problem that for every oriented graph G there exist a vertex whose out-degree at least doubles when you square the oriented graph; that is, if $G = (V(G), A(G))$ and denote $\delta^+_G(x)$ to be the out-degree of vertex x in the graph G, then there exists $x \in (G)$ such that $\delta^+_{(G^2)}(x) \geq 2\delta^+_G(x)$. This problem is also listed in the open problems in the webpage of the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS - http://dimacs.rutgers.edu/ hochberg/undopen /graphtheory /graphtheory.html).

We verify this conjecture for some families of graphs namely paths, cycles and star graphs. Moreover, we identify which of the vertices in the graph satisfies the assertion in the conjecture for most cases of the orientations of path, cycle and star graphs.

The general idea is to identify the possible orientations of the said families of graphs and from those feasible orientations analyse the neighbourhoods of each vertex. We also relate the second neighbourhood conjecture to the square of an oriented graph conjecture, that is, if the second neighbourhood conjecture is true then it must be the case that the square of an oriented graph conjecture must be true.

Keywords: tree; star chromatic index; star edge coloring

1. INTRODUCTION

In 1993, Paul Seymour formulated a conjecture regarding the square of an oriented graph, $D = (V, A)$. He defined the square of D denoted by D^2 as the digraph having the same vertex set, $V(D)$, and an arc set, $A(D^2)$ containing all arcs $(a, b) \in A$ as well as the arc (a, c) for every pair of arcs, (a, b) and $(b, c) \in A(D)$. The conjecture proposed by Seymour states that for every oriented graph D has a vertex whose out-degree in D^2 is at least twice its outdegree in D [6].

In 1995, Dean and Latka showed that Seymour's Second Neighborhood Conjecture (SNC) to be true for regular tournaments, almost regular tournaments and tournaments with minimum outdegree ≤ 5 [9].

In 1996, Fisher solved Dean's claim and he proposed Second Neighborhood Conjecture/SNC for tournaments. We say that an oriented graph D has the SNP(Second Neighborhood Property) if there exist $v \in V(D)$ such that $|N^{++}(v)| \geq |N^+(v)|$. Dean conjecture that every tournament has a vertex with the SNP. Seymour conjecture a more general statement of SNP [7].

In 2000, Havet and Thomasse showed another proof of Dean's claim. They used a tool called the median orders. They proved that a tournament that has no sink then there are at least two vertices with the second neighborhood property [7].

In 2001, Kaneko and Locke showed that Seymour's conjecture is true if the minimum out-degree of vertices in D is less than seven. [9].

In 2003, Chen, Shen and Yuster showed that in every orientation there is a vertex such that $|N^{++}(v)| \geq \gamma|N^+(v)|$ where $\gamma \approx \frac{2}{3}$ [7].

In 2007, Fidler and Yuster proved that Seymour's conjecture holds for graphs with minimum out-degree $|V(D)| - 2$, tournaments minus star and tournaments minus subtournaments. They also used median orders to show that SNC holds for some classes of digraphs [7].

In 2009, Cohn, Wright and Godbolen showed that Kaneko and Locke's work holds for random graphs almost always [9].

In the same year, Brantner, Brockman, Kay and Snively study the implications of Seymour's conjecture being false. They presented different properties as a counterexample to SNC [10].

2. PRELIMINARY

An **undirected graph** $G = (V, E)$ consists of a nonempty finite set $V = V(G)$ of elements called **vertices** and a finite set $E = E(G)$ of unordered pairs of distinct vertices called **edges**. We call $V(G)$ the **vertex set** and $E(G)$ the **edge set** of G. An edge in $E(G)$ is denoted by $\{x, y\}$. If $\{x, y\} \in E(G)$, we say that the vertices

x and y are adjacent. A **loop** is an edge that joins a single endpoint to itself. A **multiple edge** is a collection of two or more edges having identical endpoints. A **simple graph** is a graph that has no loops or multiple edges. A **walk** of length n in an undirected graph G is a non-empty alternating sequence $x_0e_0x_1e_1...e_{n-1}x_n$ of vertices and edges in G such that $e_i = \{x_i, x_{i+1}\}$ for all $i < n$. If the vertices in a walk are all distinct, it is said to be a **path** in G. If $x_0 = x_n$, it is said to be a **closed walk**. The length of any shortest path between x and y of a connected graph G is called the **distance** between x and y and is denoted $d(x, y)$. When the graph G has to be specified we use the notation $d_G(x, y)$. The graph with $n+1$ vertices $x_0, x_1, ..., x_n$ and edges $\{x_0, x_1\}, \{x_1, x_2\}, ..., \{x_{n-1}, x_n\}$ is called a **path of length n**, denoted by P_n. The **cycle of length n**, denoted by C_n is the graph with n vertices $x_0, x_1, ..., x_{n-1}$ and the edges $\{x_0, x_1\}, \{x_1, x_2\}, ..., \{x_{n-1}, x_0\}$. A **tree** is a graph with no cycle. A **star** graph or a star on n vertices is tree with one vertex adjacent to the other $n-1$ vertices.

A **directed graph** (or **digraph**) $D = (V, A)$ consists of a non-empty finite set $V = V(D)$ of elements called **vertices** and a finite set $A = A(D)$ of ordered pairs of distinct vertices called **arcs**. We call $V(D)$ the **vertex set** and $A(D)$ the **arc set** of D. An arc in $A(D)$ is denoted by (u, v). If (u, v) is an arc, we say that u is **adjacent to** v and v is **adjacent from** u.The number of arcs adjacent to vertex u is the **in-degree** of u denoted by $\delta_D^-(u)$ and the number of arcs adjacent from u is the **out-degree** of u denoted by $\delta_D^+(u)$. A vertex is said to be **sink** if it the out-degree is 0. A vertex is said to be **source** if its in-degree is 0. A vertex v is said to be a carrier if its in-degree is equal to 1 and its out-degree is also equal to 1.

An **orientation** of $G = (V, E)$ is a function $f : E \to V \times V$ such that, for all $e = \{x, y\} \in E(G)$, $f(e)$ is one of (x, y) or (y, x). An **oriented graph** is a digraph obtained by choosing an orientation for each edge of an undirected simple graph. Thus, an oriented graph can not have both oppositely directed arcs between any pair of vertices.

The **adjacency matrix** of a digraph D denoted by $\mathcal{A}(D)$ with vertex set $\{1, 2, ., n\}$ is the $n \times n$ binary matrix $\mathcal{A} = [a_{uv}]$ in which $a_{uv} = 1$ if and only if there is an arc from vertex u to v in $A(D)$.

Definition 2.1. Let $D = (V, A)$ be an oriented graph with vertex set $V(D)$ and arc set $A(D)$. The **square of an oriented graph** is a graph denoted by D^2 whose vertex set $V(D^2)$ is the same as the vertex set $V(D)$. An ordered pair of vertices (u, v) is in the arc set $A(D^2)$ if and only if either there exists a w in D such that (u, w) and (w, v) are arcs in $A(D)$ or (u, v) in $A(D)$.

Remark 2.1. *The adjacency matrix of the square of an oriented graph denoted by* $\mathcal{A}(D^2) = [\alpha_{u,v}]$ *for every* $u, v \in V(D^2)$ *where*

$$\alpha_{u,v} = \begin{cases} 1 & \textit{if}\,(u,v) \in A(D)\,\textit{or} \\ & \exists z \in V(D) \\ & \textit{such that}\,(u,z)\,\textit{and} \\ & (z,v) \in A(D) \\ 0 & \textit{otherwise} \end{cases}$$

Definition 2.2. Let D be an oriented graph. The first out-neighbor of $u \in V(D)$ denoted by $N_D^+(u)$ is defined by $N_D^+(u) = \{v \in V(D) | (u, v) \in A(D)\}$. In other words, the cardinality of $N_D^+(u)$ is equal to $\delta_D^+(u)$ is the number of arcs that is adjacent from the vertex u.

Definition 2.3. Let D be an oriented graph. The second out-neighbor of $u \in V(D)$ denoted by $N_D^{++}(u)$ are the vertices in D whose distance is two from the vertex u. The cardinality of $N_D^{++}(u)$ is denoted by $\delta_D^{++}(u)$.

Conjecture 2.1. *From [3], Seymour's Second Neighborhood Conjecture states that every oriented graph* $D = (V, A)$ *has a vertex* u *such that* $|N_D^+(u)| \leq |N_D^{++}(u)|$.

Conjecture 2.2. *(Square of Oriented Graph Conjecture) [13]*
For every oriented graph D*, there exists a vertex whose out-degree at least doubles when you square the oriented graph. This implies that there exist a vertex a vertex* $u \in D$ *such that* $\delta_{D^2}^+(u) \geq 2\delta_D^+(u)$.

3. MAIN RESULTS

Lemma 3.1. *If* D *is an oriented graph such that* $\delta_D^+(v) = 0$ *for some* $v \in V(D)$*, then the square of oriented graph conjecture is satisfied by the vertex* v.

Proof. Let $D = (V, A)$ be an oriented graph and v be a vertex $\in V(D)$ whose out-degree is 0, that is, $\delta_D^+(v) = 0$. We want to show that there exists a vertex in $V(D^2)$ such that its out-degree in D^2 at least doubles. We claim that this vertex is v. Suppose $\delta_{D^2}^+(v) \geq 1$. This implies that there is at least one arc from v to some vertex in $V(D^2)$

say u. Since there is no arc from v to any vertex u in $V(D)$, then, there should be an arc such that (v, w) and (w, u) are arcs in D. But, this is a contradiction. Hence, $\delta^+_{D^2}(v) = 0$. Therefore, the Square of Oriented Graph Conjecture is satisfied. □

Example 3.1. Consider the digraph K of an oriented path given below.

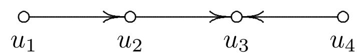

Fig. 1: The digraph K.

The square of K is given below.

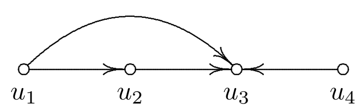

Fig. 2: The digraph K^2.

We note that $\delta^+_K(u_3) = 0$. By Lemma 3.1 the square of oriented graph conjecture is satified by the vertex u_3. The adjacency matrix of K is

$$\mathcal{A}(K) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The adjacency matrix of K^2 is

$$\mathcal{A}(K^2) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Lemma 3.2. *Let $D = (V, A)$ be an oriented graph. Then for every $u \in V(D)$, we have $|N^+_{D^2}(u)| = |N^+_D(u)| + |N^{++}_D(u)|$.*

Proof. This follows from the definition of the square of an oriented graph.Note that the first out-neighbor of u denoted by $N^+_D(u)$ cannot have a common element in the second out-neighbor of u denoted by $N^+_D(u)$.Thus, by the definition of the arc set of D^2 we have, $|N^+_{D^2}(u)| = |N^+_D(u)| + |N^{++}_D(u)|$. □

As an illustration of Lemma 3.2, consider the digraph D and its square.

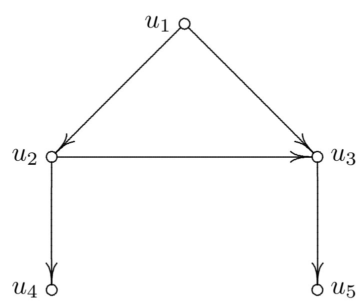

Fig. 3: The digraph D.

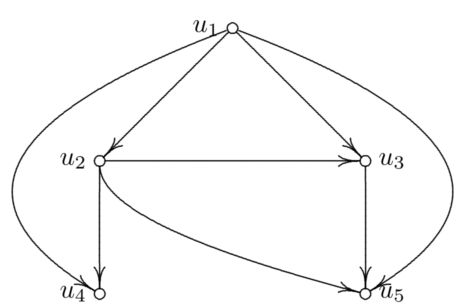

Fig. 4: The digraph D^2.

The table below shows the out-degree of each vertex in D and D^2.

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	δ^+_D	δ^{++}_D	$\delta^+_{D^2}$
u_1	2	2	4
u_2	2	1	3
u_3	1	0	1
u_4	0	0	0
u_5	0	0	0

Table I: Out-degree of vertices in D and D^2.

From Table 1, we can see that every vertex in $V(D)$ satisfies the condition for Lemma 3.2.

In the following lemma, we will verify the square of oriented graph conjecture for any orientation of stars, paths and cycles.

We now consider the orientation of star.

Lemma 3.3. *If S_n is an oriented star of length n then there exist a vertex u such that $\delta^+_{S_n^2}(u) \geq 2\delta^+_{S_n}(u)$.*

Proof. Consider the graph S_n as shown below.

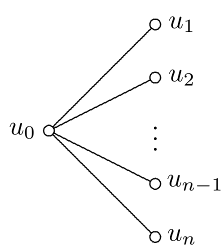

Fig. 5: The graph S_n.

In any orientation of S_n the vertices $u_1, u_2, ..., u_n$ is either a sink or a source, that is there exist a vertex in the vertex set of S_n whose out-degree is zero. Thus by Lemma 3.1, the Square of Oriented Graph Conjecture is satisfied. □

Remark 3.1. *Let S be an orientation of S_n with vertex set $V(S) = \{u_0, u_1, ..., u_n\}$ and arc set $A(S) = \{(u_j, u_0) : j \in I_1\} \cup \{(u_0, u_i) : i \in I_2\}$ where I_1 and I_2 be a partition of vertices $\{u_1, u_2, ..., u_n\}$ such that I_1 is the set of sources and I_2 is the set of sinks. The graph of an oriented star is given below.*

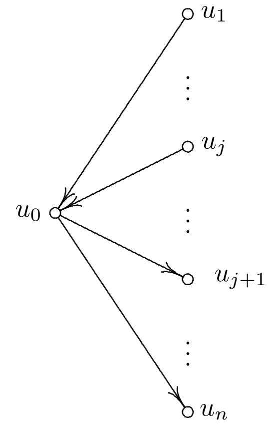

Fig. 6: The digraph S.

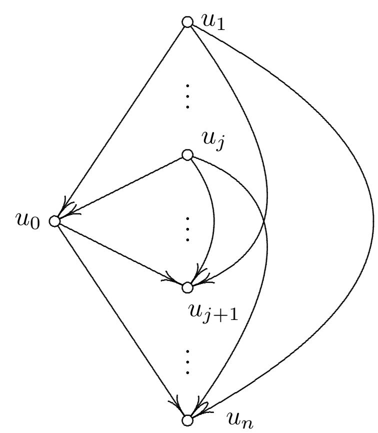

Fig. 7: The digraph S^2.

We note that we can arrange the $u_1, u_2, ..., u_j$ are sources and $u_{j+1}, u_{j+2}, ..., u_n$ are sinks. The vertices $u_1, u_2, ..., u_j$ satisfies the assertion of the square of oriented graph conjecture that is $\delta^+_{S^2}(u_i) \geq 2\delta^+_S(u_i)$ where $1 \leq i \leq j$.

We now consider some orientations of paths.

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Example 3.2. Consider the digraph B_1 of an oriented path, its square in Figures (8) and (9) and their corresponding adjacency matrices shown below.

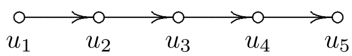

Fig. 8: The digraph B_1.

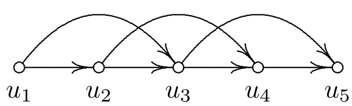

Fig. 9: The digraph B_1^2.

$$\mathcal{A}(B_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{A}(B_1^2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3.3. Now, consider the digraph B_2 of an oriented path given below.

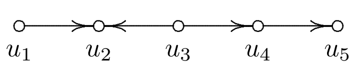

Fig. 10: The digraph B_2.

The square of B_2 and its adjacency matrix and the adjacency matrix of B_2^2 is given below.

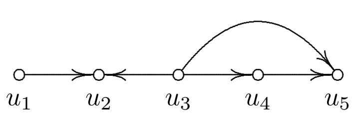

Fig. 11: The digraph B_2^2.

$$\mathcal{A}(B_2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{A}(B_2^2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the provided illustrations and examples, we now generalize the observations to a path of length n and conclude the following lemma.

Lemma 3.4. *If P_n is an oriented path of length n then there exist a vertex u such that $\delta^+_{P_n^2}(u) \geq 2\delta^+_{P_n}(u)$.*

Proof. Consider the graph P_n as shown below.

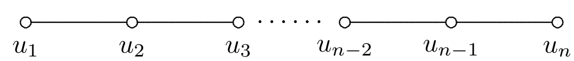

Fig. 12: The graph P_n.

In any orientation of P_n, at least one of the vertices $u_1, u_2, ..., u_n$ is either a sink or a source, that is there exist a vertex in P_n whose out-degree is zero. Thus by Lemma 3.1, the square of oriented graph conjecture is satisfied. □

Remark 3.2. *Consider another orientation of path as shown below.*

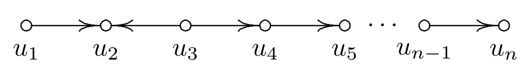

Fig. 13: The digraph P.

The square of P is given below.

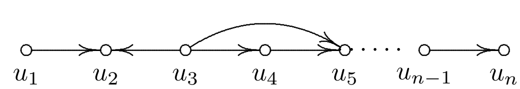

Fig. 14: The digraph P^2.

Note that u_1 and u_3 are sources, u_2 and u_n are sinks while $u_4, ..., u_{n-1}$ are carriers. The vertices $u_3, ..., u_{n-2}$ satisfies the assertion of the square of oriented graph conjecture that is $\delta^+_{P^2}(u_i) \geq 2\delta^+_{P}(u_i)$ where $3 \leq i \leq n-2$.

We now consider some orientations of cycle.

Example 3.4. Consider the digraph H_1 of an oriented cycle given below.

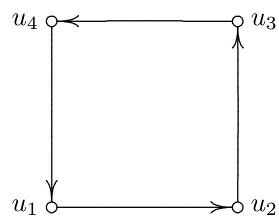

Fig. 15: The digraph H_1.

The square of H_1 is given below.

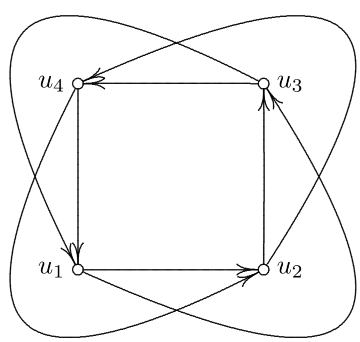

Fig. 16: The digraph H_1^2.

The adjacency matrix of H_1 is

$$\mathcal{A}(H_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The adjacency matrix H_1^2 is

$$\mathcal{A}(H_1^2) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

.

Example 3.5. Now, consider the digraph H_2 of an oriented cycle given below.

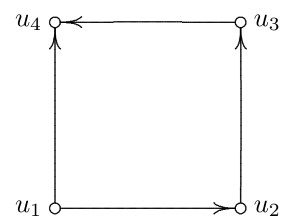

Fig. 17: The digraph H_2.

The square of H_2 is given below.

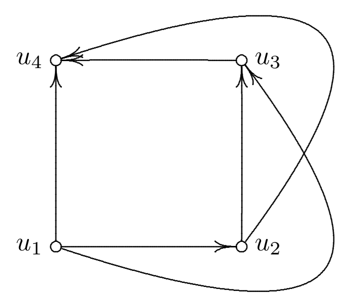

Fig. 18: The digraph H_2^2.

The adjacency matrix of H_2 is

$$\mathcal{A}(H_2) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The adjacency matrix H_2^2 is

$$\mathcal{A}(H_2^2) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

.

From the provided illustrations and examples, we now generalize the observations to a cycle of length n and conclude the following lemma.

Lemma 3.5. *If C_n is an oriented cycle of length n then there exist a vertex u such that $\delta^+_{C_n^2}(u) \geq 2\delta^+_{C_n}(u)$.*

Proof. Consider the graph C_n as shown below.

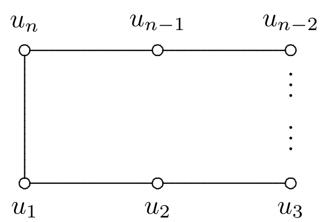

Fig. 19: The graph C_n.

The vertices $u_1, u_2, ..., u_n$ is either a sink or a source, that is there exist a vertex in C_n whose out-degree is zero. Thus by lemma 3.1, the square of oriented graph conjecture is true. □

Remark 3.3. *Consider another orientation of cycle as shown below.*

The square of C is given below.

Note that u_1 is a source, u_n is a sink and $u_2, u_3, ..., u_{n-1}$ are carriers. The vertices $u_2, u_3, ..., u_{n-2}$ satifies the square of oriented graph conjecture that is $\delta^+_{C^2}(u_i) \geq 2\delta^+_C(u_i)$ where $2 \leq i \leq n-2$.

We now define what an acyclic digraph is and show a theorem about it. Acyclic graph is a graph with no cycles [15]. An acyclic orientation of G is one in which there are no directed cycles. An acyclic digraph D is an

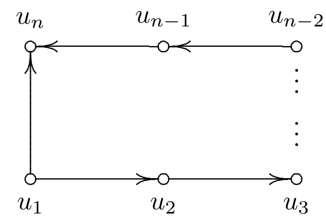

Fig. 20: The digraph C.

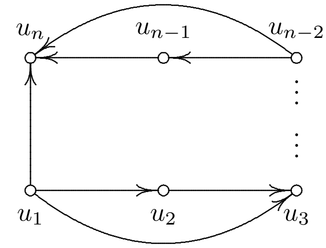

Fig. 21: The digraph C^2.

oriented graph wherein there are no directed cycle, no multiple edge and no loops [8]. A known theorem about acyclic digraph is shown below.

Theorem 3.6. *[5] An acyclic digraph has at least one point of outdegree zero.*

Note: For every acyclic digraph the square of oriented graph conjecture is trivially satisfied. This is also true from Lemma 3.1, since there is a vertex whose out-degree is zero then the square of oriented graph conjecture is also satisfied.

Theorem 3.7. *Let D be an oriented graph. If Seymour's second neighborhood conjecture is true, then the square of oriented graph conjecture is true.*

Proof. Let $D = (V, A)$ be an oriented graph. If the Seymour's Second Neighbourhood Conjectire is true, then $\exists x \in V(D)$ such that $|N^{++}_D(x)| > |N^+_D(x)|$. Let $\delta^+_D(x) = m$ then we have $|N^{++}_D(x)| > m$. From Lemma 3.2, we can see that $\delta^+_{D^2}(x) =\geq 2m$.

Thus $\delta^+_{D^2}(x) \geq 2\delta^+_D(x)$. Therefore the assertion of the theorem follows.

□

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4. SUMMARY, CONCLUSION AND RECOMMENDATION

We have verified and shown all the possible orientations for the families of graphs which includes path, cycle and star that will satisfy the square of oriented graph conjecture.

Based from this study, we can now say that some orientation of paths, cycles and star graphs satisfied the square of oriented graph conjcture and we can also say that if the second neighbourhood conjecture is true, then the square of oriented graph conjecture is also true.
We further recommend other researchers to verify the conjecture for other families of graphs.

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