Some Notes on Harmonious Labeling of Friendship Graphs F_n

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Abstract: In the course MTH 971D (Selected Topics in Mathematics (Graph Theory)) during the 1st Term 2015-16, students were required to make an exposition of a recently published paper. The work of Tanna (2013) involved reiterations of proofs, as well as, supplementary examples to an earlier work of Graham and Sloane (1980) concerning harmonious labeling of certain classes of graphs. This student found out that there were some clarificatory findings in the harmonious labeling of friendship graphs F_n.

In this paper, focus will be given on the result "the friendship graph F_n is harmonious except for $n \equiv 2 \bmod 4$". It was noted that: (1) there was a need to supplement with a fourth case the proof of this theorem (both papers cited only three cases in their proofs); (2) additional supplementary examples, as well as illustrations on how to generate the labels for F_4 and F_5 using the algorithm of Skolem (1957), will be presented; and, (3) a "counterexample" as to why the case for harmoniously labeling $F_n (n \equiv 3 \bmod 4, n \geq 7, n \in N)$ fails for F_7.

Key Words: Harmonious labeling; friendship graph; congruent to (congruence)

1. Introduction

In the course MTH 971D (Selected Topics in Mathematics (Graph Theory)) during the 1st Term 2015-16, students were required to make an exposition of a recently published paper.

A particular topic of interest was on labeling of graphs – specifically, on harmoniously labeling graphs. The work of Tanna (2013) involved reiterations of proofs, as well as, supplementary examples to an earlier work of Graham and Sloane (1980) concerning harmonious labeling of certain classes of graphs.

The study of graph labelling is very much useful since its applications can be found in coding theory, astronomy, and circuit design to name a few.

This study aimed to supplement the proof of the mentioned works as well as give additional illustrations on how to harmoniously label specific friendship graphs F_4 and F_5.

2. Preliminaries

2.1 Some Basic Notions on Graphs

Graph Theory is the study of *graphs* – mathematical relations used to model pairwise relations between objects. A ***graph*** is an ordered triple $G = (V(G), E(G), I_G)$ of the nonempty set ***V(G)*** called the ***vertex*** (***node*** or ***point***) ***set*** of G, the set ***E(G)*** called the ***edge*** (***arc*** or ***line*** or ***link***) ***set***, and an ***incidence*** relation I_G associating each element of $E(G)$ to an unordered pair of (same or distinct) elements of *V*(*G*). If ***e*** is an edge of *G* and $I_G(e) = \{u, v\}$, where ***u*** and ***v*** are vertices in *V*(*G*), we write $e = uv$. Here, we say *u* and *v* are ***adjacent*** to each other, while *e* is said to be ***incident*** to both *u* and *v* or *v* and *u* are ***incident*** to ***e***.

To illustrate:

Consider the graph $G = (V(G), E(G), I_G)$, where $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$, and

$$I_G : e_1 = v_1v_5,\ e_2 = v_2v_3,\ e_3 = v_2v_5,\ e_4 = v_2v_5,\ e_5 = v_3v_3.$$

(see corresponding graph in Figure 1).

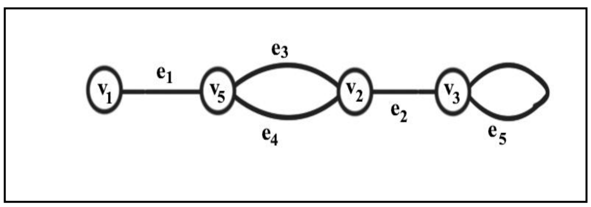

Fig.1. The graph of G as given in the above illustration

The given graph in Figure 1 is an example of graph that which is a ***multigraph*** since (1) it has two vertices v_2 and v_5 with two edges e_3 and e_4 incident to them, and (2) it has a ***loop*** as edge e_5 is incident only to one vertex e_3.

Most of the time, the focus of the study of graphs is about ***simple graphs*** – those with no multiple edges or loops. An illustration of a simple graph is given in Figure 2.

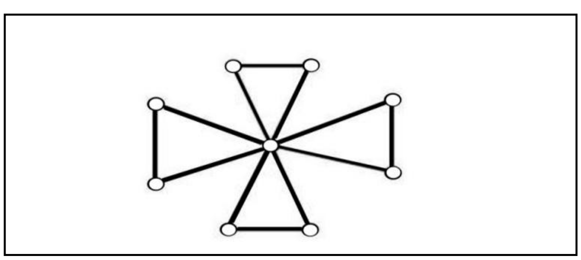

Fig. 2. An example of a simple graph with 9 vertices and 12 edges

Since this paper is on harmonious labelling of graphs, it is but apt to define what a labeled graph is.

A graph G is said to be ***labeled*** if its n vertices are distinguished from one another by labels such as $v_1, v_2, v_3, \ldots, v_n$. Unless otherwise stated, it is assumed in most labelings that set of natural numbers is the source of the vertex labels. Furthermore, there are graph labelings that need to satisfy the following characteristics: (1) a set of numbers from which the vertex labels are chosen; (2) a rule that assigns a value to each edge; and, (3) a condition that these values in (1) and/or (2) must satisfy. Figure 3 shows a vertex labeling of six vertices labeled, while the graph in Figure 4 shows both vertex and edge labeling of a graph with 16 vertices and 22 edges.

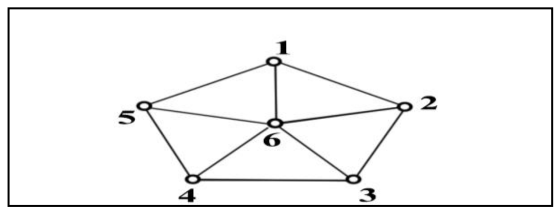

Fig. 3. A vertex labeling of a graph with six vertices

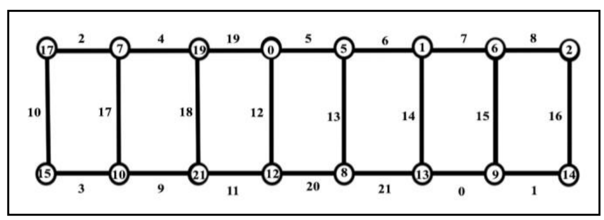

Fig. 4. An example of a graph with both vertex and edge labeling

Graph labelings trace their origins from Rosa's *On Certain Valuation of the Vertices of a Graph* (1967) where he discussed certain properties when a graph has an $\alpha-$valuation, a $\beta-$valuation, a $\sigma-$valuation, and a $\rho-$ valuation.

Various types of vertex labelings of graphs were enumerated by Serra (2009) at CIMPA-Indonesia are *graceful*, *bigraceful*, *harmonious*, *cordial*, *equitable*, hamming *graceful*, etc. Additionally, he also gave a listing of edge labelings and total labelings of graphs such as *magic*, *supermagic, antimagic,* etc.

2.2 Some Special Terms Needed

The ***order of a graph G*** is its number of vertices. This is denoted by either $|V(G)|$ or $n(G)$.

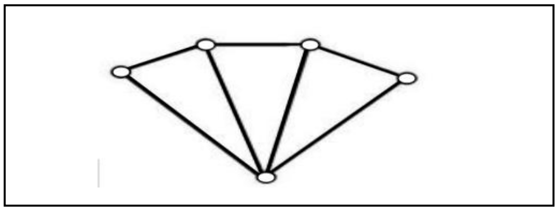

Fig. 5. A graph G with $|V(G)|=5$

A ***path*** P_n is a graph that is an alternating sequence $v_1e_1v_2e_2v_3e_3...v_{n-1}e_{n-1}v_n$ of vertices and edges that starts and ends with a vertex, in which

$I_{P_n}(e_i)=v_iv_{i+1}$ where $i=1,2,...,n-1$.

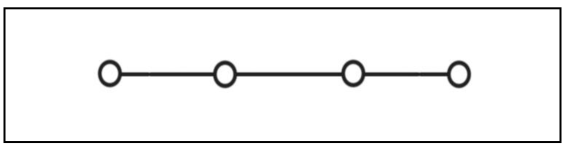

Fig. 6. A path P_4

A ***cycle*** C_n is a graph that is an alternating sequence $v_1e_1v_2e_2v_3e_3...v_{n-1}e_{n-1}v_ne_nv_1$ of vertices and edges that starts and ends with a vertex, in which

$$I_{C_n}(e_i)=\begin{cases} v_iv_{i+1} & \text{if } \quad i=1,2,...,n-1 \\ v_iv_{i-n} & \text{if } \quad i=n \end{cases}.$$

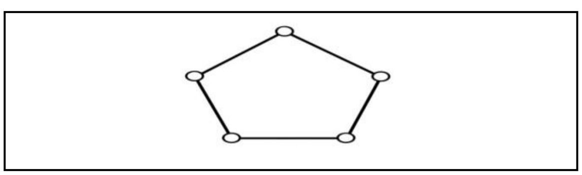

Fig. 7. A cycle C_5

The ***Friendship Graph*** F_n is a graph that consists of n triangles with a common vertex. An illustration of a friendship graph appears in Figure 2.

Consider a graph $G=(V(G),E(G),I_G)$ with k edges. A function f, defined by $f:V(G)\to\{1,2,3,...,k-1\}$, is called a ***harmonious labeling of G*** if it is injective and it induces a bijective function f^* defined by

$f^*(e)=(f(u)+f(v))\bmod k$, where $e=uv$ for $u,v\in V(G), e\in E(G)$.

A graph satisfying the conditions of a harmonious labeling is called a ***harmonious graph***. Figures 8 and 9 show two ways of labelling the complete graph K_4 .

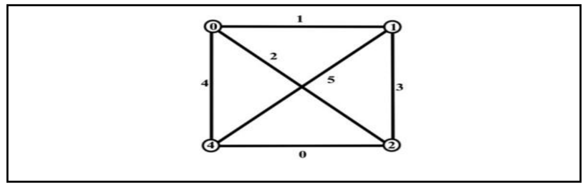

Fig. 8. An illustration of a harmonious labelling of K_4

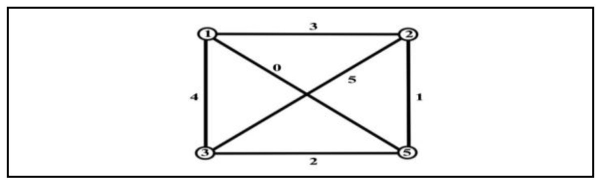

Fig. 9. Another way of harmoniously labelling K_4

3. Methodology

The paper of Tanna (2013) was reviewed and the proofs were analyzed using the definitions relevant to the friendship graph and harmonious labeling. The list of references were also obtained and upon seeing that Graham and Sloane (1980) had already stated and proven the result of harmoniously labeling graphs, their work were also reviewed to come up with a more comprehensive report as required for MTH 971D. Modifications and/or improvement of the proof, as well as additional examples on how to label specific friendship graphs were obtained using the algorithm mentioned by Skolem (1957) as cited by both works.

4. Results

Both the works of Tanna (2013) and Graham and Sloane presented the proof for the following theorem " The Friendship Graph F_n is harmonious except for $n\equiv 2 \bmod 4$." using the following cases

Case 1. $n\equiv 2 \bmod 4$

Case 2. $n\equiv 0 \bmod 4$ or $n\equiv 1 \bmod 4$

Case 3. $n \equiv 3 \bmod 4 \quad (n \geq 7, n \in N)$

But it should be noted that in all the mentioned cases, the case for harmoniously labeling F_3 was not taken in consideration since the proof of case 3 restricts the value of n , which was added by this author since both required to consider the set $\{1,2,\ldots,2n-6\}$ and use the partition of $n-3$ pairs of the form (a_r, b_r) with $b_r - a_r = r+2$ where $r = 1,2,\ldots,n-3$ in order to determine a harmonious labeling of the vertices of the cycles by

$(0,1,3n-1),(0,2,3n-6),(0,3n-2,3n-3),$ and $(0,r+2,n+a_r)$ with 0 as the label assigned to the common vertex. However, 3 cannot satisfy this as r will range from 1 to 0 (an impossibility!), and thus there must be a need for

Case 4. A harmonious labeling of F_3 .

Such labeling can be obtained by labeling the triangles with the following vertices $(0,1,8),(0,3,4),$ and $(0,5,6)$ with the harmonious graph given in Figure 10.

It should be noted that the labelling induced in Figure 10 satisfied the definition of a harmonious labeling – an injective mapping making use of a subset of the edge set and an induced bijective mapping for the nine edges from 0 to 9.

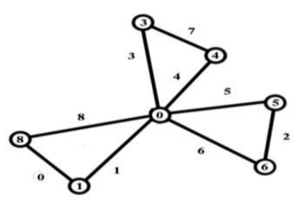

Fig. 10. Harmonious graph F_3

Tanna (2013) and Graham and Sloane(1980) also made use of the algorithm of Skolem (1957) in proving case 2. Skolem (1957) showed that set of numbers in the set of ordered pairs (a_r, b_r) can be distributed in n pairs with $b_r - a_r = r$ where $r = 1,2,\ldots,n$. Both papers on harmonious labeling of friendship graphs mentioned that a harmonious graph can be obtained by using the recursive equation as given, by labeling the vertices of the cycles in the friendship graph by $(0,r,n+a_r)$, where $r = 1,2,\ldots,n$ and 0 is the label assigned to the common vertex.

In lieu of this, Tanna (2013) illustrated a harmonious labelling satisfying this case using the graphs of F_4 and F_5 as given in Figures 11 and 12, respectively.

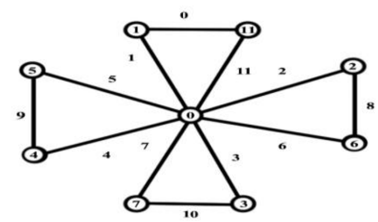

Fig. 11. A harmonious labelling of F_4 as illustrated by Tanna (2013)

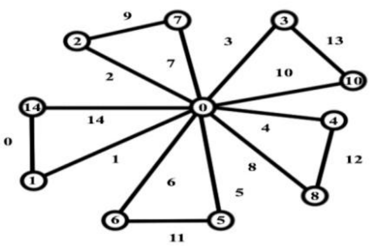

Fig. 12. A harmonious labelling of F_5 as given by Tanna (2013).

Now, the details on where the labels for these vertices came from were not fully disclosed in Tanna's work. This author gives the following table of computations (see Table 4 and 5) from which the labels for F_4 and F_5 were obtained. Following from Skolem's algorithm and the labelling as suggested by both Graham and Sloane (1980) and Tanna (2013), consider subdividing the set $S = \{1,2,3,4,5,6,7,8\}$ into 4 ordered pairs (a_r, b_r) satisfying the table of values below. The table will help in generating the following harmonious graph, with vertices $(0,r,n+a_r)$.

However, Tanna remarked in his work that harmonious labelings of graphs need not be unique. This led for this author to discover using Skolem's algorithm an alternative list of values for $b_r, a_r,$ and r in generating another set of harmonious labelings of F_4 and F_5 , as given below. Another labelling of F_4 can be generated using the following table with its graph given in Figure 13. For F_5 , consider Table 4 and its corresponding harmonious graph on Figure 14.

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Table 1. List of b_r, a_r, and r that will be used to generate the labeling of F_4 in Tanna's work.

b_r	a_r	r
8	7	1
4	2	2
6	3	3
5	1	4

Table 2. Values of b_r, a_r, and r to generate the labeling of F_5.

b_r	a_r	r
10	9	1
4	2	2
8	5	3
7	3	4
6	1	5

Table 3. Another list of values for b_r, a_r, and r to generate the labeling of F_4.

b_r	a_r	r
8	7	1
5	3	2
4	1	3
6	2	4

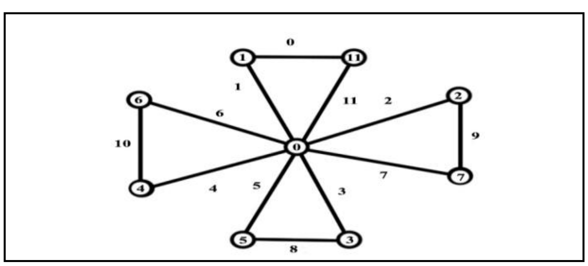

Figure 13. Another harmonious labelling of F_4 using Table 3 as reference

Table 4. Another set of values of b_r, a_r, and r to generate the labeling of F_5.

b_r	a_r	r
4	3	1
10	8	2
9	6	3
5	1	4
7	2	5

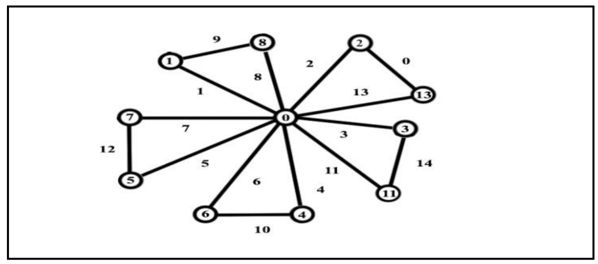

Figure 14. A harmonious labelling of F_5 using the Table 4.

Finally, it was also noted that there seemed to be a particular case that will not be satisfied by case 3: $F_n \quad (n \equiv 3 \bmod 4; n \geq 7, n \in N)$.

Again, both the works of Tanna (2013) and Graham and Sloane (1980) suggested that the set $\{1,2,...,2n-6\}$ be considered and a partition of $n-3$ pairs of the form (a_r, b_r) with $b_r - a_r = r+2$ where $r = 1,2,...,n-3$ be used to generate the harmonious labeling for case 3 for which F_7 supposedly should have satisfied as given in both papers. However, upon generating combinations of values for b_r, a_r and r, it can be seen from the following tables that the formula mentioned will not hold for F_7.

Table 5. List of possible combinations of of b_r, a_r, and r in order to generate the labeling of F_7.

r	$r+2$	(a_r, b_r)
4	6	$(1,7),(2,8)$
3	5	$(1,6),(2,7),(3,8)$
2	4	$(4,8),(2,6),(1,5),(3,7)$
1	2	$(5,8),(4,7),(3,6),(2,5),(1,4)$

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Table 6. Incomplete set of values of $b_r, a_r,$ and r to generate the labeling of F_7.

b_r	a_r	$r+2$
??	??	3
7	3	4
6	1	5
8	2	6

Table 7. Incomplete set of values of $b_r, a_r,$ and r to generate the labeling of F_7

b_r	a_r	$r+2$
??	??	3
6	2	4
8	3	5
7	1	6

Tables 5-7 show the argument as to why the proof of case 3 for F_n . The second column of Table 5 shows the only possible pairs that will generate the difference $b_r - a_r = r$. Since the least possible source that gives a $b_r - a_r = 6$, then $(1,7)$ and $(2,8)$ were used to start filling in Tables 6 and 7. After which we generate the next value $b_r - a_r = 5$, with the restriction in mind that the values 8 and 2 should not appear for Table 6 in the row being considered, while the number 7 and 1 should not be seen in the ordered pair for the row $r+2=5$. The process is just continued with the same principle applied, but when it came to finding the entries for the row when $b_r - a_r = r+2 = 3$, the only un-utilized values for both tables are 5 and 4, which means that these will not give the desired difference . Therefore, the property for case 3 fails when using F_7 .

5. Summary

In dealing with articles published as scholarly output, one needs to be critical as to how the paper was presented and consequently the details of the proof made available to the layperson. It just so happened that both papers considered in this study, needed improvements so as to facilitate correctness and completeness of the proofs regarding how to harmoniously label friendship graphs.

6. Acknowledgment

I want to thank my daughter Cresel for helping me create the graphs that were constructed in the figures.

7. References

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