Eigenvalues of Cross and Box Products of Cycles C_n, $n = 3,4,5$ and Paths P_m, $m = 2,3,4$

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Abstract: In this study, the cross products and box products, which are two operations defined by Kilp and Knauer (2001) in their work *Graph Operations and Categorical Constructions*, were considered. These operations were applied to cycles C_n, $n = 3, 4, 5$ and paths P_m, $m = 2, 3, 4$. The adjacency matrices of the resulting graphs were constructed and eigenvalues of each graph were determined. It was shown that the eigenvalue of the cross product was the product of the eigenvalues of each graph, while the eigenvalue of the box product was the sum of the eigenvalues of each graph.

Key Words: Eigenvalues; cross product; box product; paths; cycles

1. Introduction

Any simple graph has a corresponding adjacency matrix and for each matrix, eigenvalues can be computed.

Applications of eigenvalues of a matrix abound in the fields of science and engineering. Some of them are in Control theory, vibration analysis, electric circuits, advanced dynamics, quantum mechanics, physics, statistics and even finance.

Many studies have been done involving eigenvalues of a matrix but very few studies could be found in the literature regarding cross and box products. Most studies that are related to this paper are on nullity and rank of a matrix. One study on nullity of graphs was presented by Bo Cheng and Bolian Liu in their paper entitled *On the Nullity of Graphs* published in the International Linear Algebra Society last January 2007. Another study conducted as undergraduate special problem in University of the Philippines Los Banos in 2009 was entitled *Graph Operations and Their Induced Autographs* by Neil Jerome Egarguin.

In 2001, Kilp and Knauer, in their work *Graph Operations and Categorical Constructions*, defined certain operations on graphs, including the cross and box products.

This study aimed to find a relationship between the eigenvalues of the adjacency matrices of cycles and paths and the eigenvalues of the adjacency matrices of their cross products and box products. Due to time limit and the limit in the capacity of the software used to find the eigenvalues of cross and box products, paths considered were only those with 2,3 and 4 vertices while cycles considered were those with 3,4 and 5 vertices.

2. Preliminaries

A ***graph*** G consists of the ***vertex set*** $V(G)$ and ***edge set*** $E(G)$. A ***cycle*** C_n is a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. A ***path*** P_n is a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ where the vertices $v_1, v_2, \ldots, v_n$ are all distinct.

Two vertices of a graph are ***adjacent*** if they are connected by an edge. A graph G, with n vertices, can be represented by its ***adjacency matrix*** $A(G)$, which is the $n \times n$ symmetric matrix

$$\left[a_{ij}\right] = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \end{cases}$$

as illustrated in Figure 1.

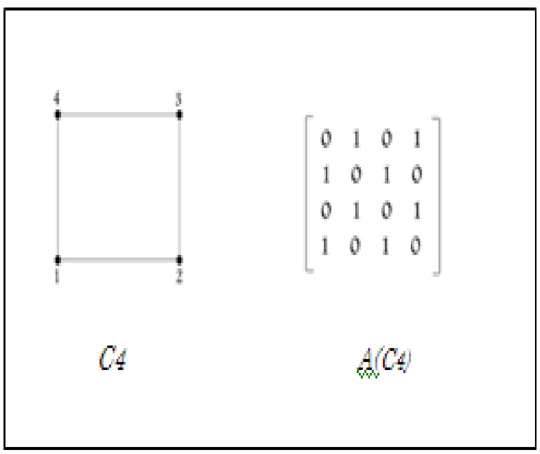

Fig. 1. The cycle C_4 and its corresponding adjacency matrix

The ***cross product*** of graphs G and H is given by $G \times H$, where the vertex set is

$$V(G \times H) = V(G) x V(H)$$

and edge set is

$$E(G \times H) = \{\left[gh, g'h'\right] / gg' \in E(G) \wedge hh' \in E(H)\}$$

As an illustration, consider the graphs G and H in Figure 2 and their cross product $G \times H$, which is actually $C_3 \times C_3$.

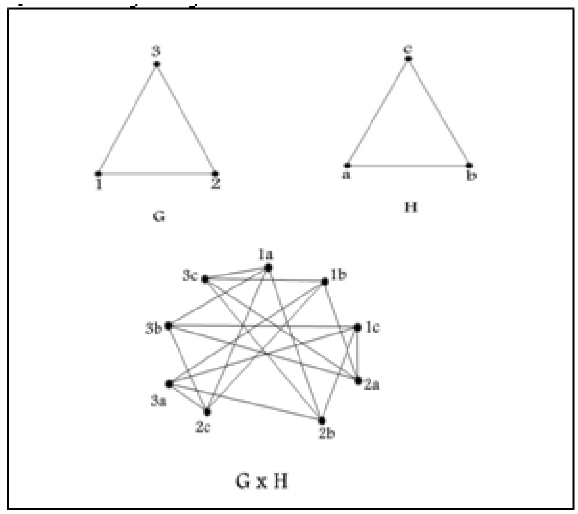

Fig. 2. The cycles G and H with 3 vertices and the corresponding cross product.

On the other hand, the ***box product*** $G \square H$, has vertex set

$$V(G \square H) = V(G) x V(H)$$

and edge set

$$E(G \square H) = \{\left[gh, g'h'\right] / (g = g' \text{ if } hh' \in E(H)) \vee (h = h' \text{ if } gg' \in E(G))\}$$

See example of $C_3 \square C_4$ in Figure 3.

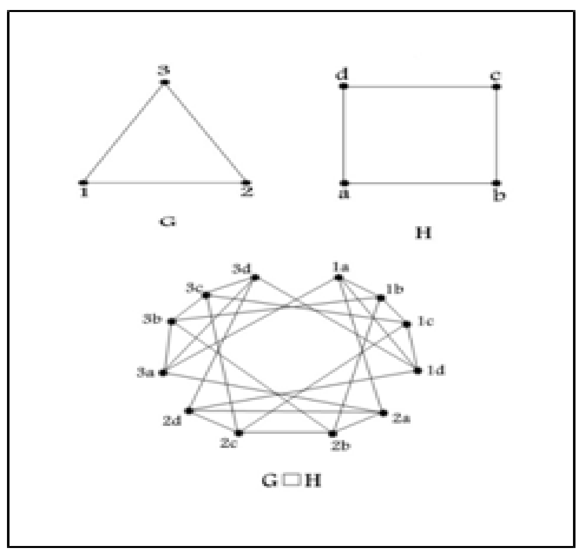

Fig. 3. The cycles with 3 and 4 vertices, respectively, and the corresponding box product

The ***eigenvalues*** $\lambda_1, \lambda_2, ..., \lambda_n$ of a graph G are the eigenvalues of its adjacency matrix $A(G)$. In 2007, Cheng and Liu were able to generate formulas for the eigenvalues of paths P_m and cycles C_n.

They derived that the eigenvalues of the adjacency matrices for paths P_m are given by

$$\lambda_r = 2\cos\left(\frac{\pi r}{m+1}\right)$$

where $r = 1,2, ..., n$, while for cycles C_n, they were able to obtain the eigenvalues to be

$$\lambda_r = 2\cos\left(\frac{2\pi r}{n}\right)$$

where $r = 0, 1, 2, 3, ..., n-1$.

3. Methodology

For the eigenvalues of cycles and paths, the formulas of Cheng and Liu were used. But for simplicity, the formula for the eigenvalues λ_r of C_n was adjusted so that $r = 1,2, ..., n$, since the same value will be obtained when $r = 0$ and when $r = n$. The formula now becomes

$$\lambda_r = 2\cos\left(\frac{2\pi r}{n}\right)$$

where $r = 1, 2, 3, ..., n$.

For the eigenvalues of the cross and box products, a software called MAGMA was used . This software generates eigenvalues as a set of ordered pairs with the eigenvalues as first components and their corresponding multiplicities as the second components.

An illustration of the computations by MAGMA can be seen in Figure 4. This shows that the eigenvalues are 0, -2 and 2 with multiplicities 2, 1 and 1 respectively.

Fig. 4. An example of a computer output for the eigenvalues of C_4 obtained using MAGMA

Now, using the adjusted formula of Cheng and Liu, it was verified below that the same eigenvalues of C_4 are obtained.

$$\lambda_1 = 2\cos\left(\frac{2\pi\ (1)}{4}\right) = 2\cos\left(\frac{\pi}{2}\right) = 2\left(0\right) = 0$$

$$\lambda_2 = 2\cos\left(\frac{2\pi\ (2)}{4}\right) = 2\cos\left(\frac{4\pi}{4}\right) = 2\cos\left(\pi\right) = 2(-1) = -2$$

$$\lambda_3 = 2\cos\left(\frac{2\pi\ (3)}{4}\right) = 2\cos\left(\frac{3\pi}{2}\right) = 2(0) = 0$$

$$\lambda_4 = 2\cos\left(\frac{2\pi\ (4)}{4}\right) = 2\cos\left(2\pi\right) = 2(1) = 2$$

After computing all the desired eigenvalues, some iterative formulas were obtained that could generate all the eigenvalues of the cross and box products in a certain order.

4. Results

The following are the results of this study. Take note that the operations cross product and box product were applied only to paths P_m (with vertices $m = 2,3,4$) and cycles C_n (with vertices $n = 3,4,5$). The method for showing these properties was by Exhaustion.

Proposition 1. Let C_m and C_n be cycles with m,n vertices, respectively . Also, let their eigenvalues be given respectively as

$$\lambda_i' = 2\cos\left(\frac{2\pi i}{m}\right) \quad , \quad i = 1,2,\ldots,m$$

and

$$\lambda_j'' = 2\cos\left(\frac{2\pi j}{n}\right) \quad , \quad j = 1,2,\ldots,n$$

Then the eigenvalues of C_m x C_n when

Case 1. $m = n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') \bullet (\lambda_j'')$.

If $i > j$, then $\lambda_{2i+j} = (\lambda_i') \bullet (\lambda_j'')$.

Case 2. $m \ne n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') \bullet (\lambda_j'')$.

If $i > j$, then $\lambda_{2i+j} = (\lambda_i') \bullet (\lambda_j'')$.

If $i = m$ and $j = n$, then

$$\lambda_{2i+j} = (\lambda_i') \bullet (\lambda_j'')$$

Sketch of Proof: The formula will be verified for $m = n = 3$ (3 being the number of vertices of the smallest cycle). Using the formula of Cheng and Liu,

$$\lambda_i' = 2\cos\left(\frac{2\pi i}{3}\right) = 2\cos\left(\frac{2\pi j}{3}\right) = \lambda_j'' \quad ,$$

where $i = j = 1,2,3,$, we have

$$\lambda_1' = 2\cos\left(\frac{2\pi}{3}\right) = -1 = \lambda_1'' \quad ,$$

$$\lambda_2' = 2\cos\left(\frac{4\pi}{3}\right) = -1 = \lambda_2'' \quad ,$$

and

$$\lambda_3' = 2\cos\left(\frac{6\pi}{3}\right) = 2 = \lambda_3'' \quad .$$

Then for $i \le j$,

$$\lambda_{1\bullet 1} = \lambda_1 = (\lambda_1') \bullet (\lambda_1'') = (-1)(-1) = 1$$
$$\lambda_{1\bullet 2} = \lambda_2 = (\lambda_1') \bullet (\lambda_2'') = (-1)(-1) = 1$$
$$\lambda_{1\bullet 3} = \lambda_3 = (\lambda_1') \bullet (\lambda_3'') = (-1)(2) = -2$$
$$\lambda_{2\bullet 2} = \lambda_4 = (\lambda_2') \bullet (\lambda_2'') = (-1)(-1) = 1$$
$$\lambda_{2\bullet 3} = \lambda_6 = (\lambda_2') \bullet (\lambda_3'') = (-1)(2) = -2$$

and

$$\lambda_{3\bullet 3} = \lambda_9 = (\lambda_3') \bullet (\lambda_3'') = (2)(2) = 4 \; .$$

For $i > j$, with $i = 2,3$ and $j = 1,2$:

$$\lambda_{2\bullet 1} = \lambda_{2\bullet 2+1} = \lambda_5 = (\lambda_2') \bullet (\lambda_1'') = (-1)(-1) = 1$$
$$\lambda_{3\bullet 1} = \lambda_{2\bullet 3+1} = \lambda_7 = (\lambda_3') \bullet (\lambda_1'') = (2)(-1) = -2$$

and

$$\lambda_{3\bullet 2} = \lambda_{2\bullet 3+2} = \lambda_8 = (\lambda_3') \bullet (\lambda_2'') = (2)(-1) = -2$$

Thus, the eigenvalues of C_3 x C_3 are

$$\lambda_{C_3 \text{ x } C_3} = 1,\ 1,\ -2,\ 1,\ 1,\ -2,\ -2,\ -2,\ 4 \; .$$

These show that for the formula holds for $m = n = 3$

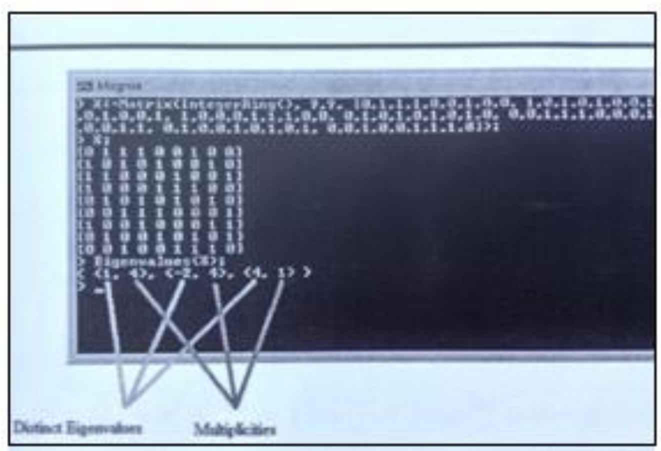

Fig. 5. The computed set of eigenvalues for C_3 x C_3 using MAGMA

The formula can be verified to hold also for C_3 x C_4, C_3 x C_5, C_4 x C_4, and C_5 x C_5 .

The rest of the Propositions can be derived in a similar manner.

Proposition 2. Let P_m and P_n be paths with m,n vertices, respectively . Also, let their eigenvalues be given respectively as

$$\lambda_i' = 2\cos\left(\frac{\pi i}{m+1}\right) \quad , \ i = 1,2,\ldots,m$$

and

$$\lambda_j'' = 2\cos\left(\frac{\pi j}{n+1}\right) \quad , \ j = 1,2,\ldots,n$$

Then the eigenvalues of P_m x P_n when

Case 1. $m = n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') \bullet (\lambda_j'')$.

If $i > j$, then $\lambda_{2i-j} = (\lambda_i') \bullet (\lambda_j'')$.

Case 2. $m \ne n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') \bullet (\lambda_j'')$.

If $i > j$, then $\lambda_{2i-j} = (\lambda_i') \bullet (\lambda_j'')$.

If $i = m$ and $j = n$, then $\lambda_{2i-j} = (\lambda_i') \bullet (\lambda_j'')$.

Proposition 3. Let C_m and P_n be cycles and paths with *m,n* vertices, respectively . Also, let their eigenvalues be given respectively as

$$\lambda_i' = 2\cos\left(\frac{2\pi i}{m}\right) \quad , \ i = 1,2,\ldots,m$$

and

$$\lambda_j'' = 2\cos\left(\frac{\pi j}{n+1}\right) \quad , \ j = 1,2,\ldots,n$$

Then the eigenvalues of C_m x P_n when

Case 1. $m = n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') \bullet (\lambda_j'')$.

If $i > j$, then $\lambda_{2i-j} = (\lambda_i') \bullet (\lambda_j'')$.

Case 2. $m \ne n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') \bullet (\lambda_j'')$.

If $i > j$, then $\lambda_{2i-j} = (\lambda_i') \bullet (\lambda_j'')$.

If $i = m$ and $j = n$, then $\lambda_{2i-j} = (\lambda_i') \bullet (\lambda_j'')$.

Proposition 4. Let C_m and C_n be cycles with *m,n* vertices, respectively . Also, let their eigenvalues be given respectively as

$$\lambda_i' = 2\cos\left(\frac{2\pi i}{m}\right) \quad , \ i = 1,2,\ldots,m$$

and

$$\lambda_j'' = 2\cos\left(\frac{2\pi j}{n}\right) \quad , \ j = 1,2,\ldots,n$$

Then the eigenvalues of $C_m \square C_n$ when

Case 1. $m = n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') + (\lambda_j'')$.

If $i > j$, then $\lambda_{2i+j} = (\lambda_i') + (\lambda_j'')$.

Case 2. $m \ne n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') + (\lambda_j'')$.

If $i > j$, then $\lambda_{2i+j} = (\lambda_i') + (\lambda_j'')$.

If $i = m$ and $j = n$, then $\lambda_{2i+j} = (\lambda_i') + (\lambda_j'')$.

Proposition 5. Let P_m and P_n be cycles with *m,n* vertices, respectively . Also, let their eigenvalues be given respectively as

$$\lambda_i' = 2\cos\left(\frac{\pi i}{m+1}\right) \quad , \ i = 1,2,\ldots,m$$

and

$$\lambda_j'' = 2\cos\left(\frac{\pi j}{n+1}\right) \quad , \ j = 1,2,\ldots,n$$

Then the eigenvalues of $P_m \square P_n$ when

Case 1. $m = n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') + (\lambda_j'')$.

If $i > j$, then $\lambda_{2i-j} = (\lambda_i') + (\lambda_j'')$.

Case 2. $m \ne n$

If $i \le j$, then $\lambda_{ij} = (\lambda_i') + (\lambda_j'')$.

If $i > j$, then $\lambda_{2i-j} = (\lambda_i') + (\lambda_j'')$.

If $i = m$ and $j = n$, then $\lambda_{2i-j} = (\lambda_i') + (\lambda_j'')$.

Proposition 6. Let C_m and P_n be cycles with *m,n* vertices, respectively . Also, let their eigenvalues be given respectively as

$$\lambda_i' = 2\cos\left(\frac{2\pi i}{m}\right) \quad , \ i = 1,2,\ldots,m$$

and

$$\lambda_j'' = 2\cos\left(\frac{\pi j}{n+1}\right) \quad , \ j = 1,2,\ldots,n$$

Then the eigenvalues of $C_m \square P_n$ when

Case 1. $m = n$

If $i \le j$, then $\lambda_{i\,j} = (\lambda_i') + (\lambda_j'')$.

If $i > j$, then $\lambda_{2\mathrm{i}-j} = (\lambda_i') + (\lambda_j'')$.

Case 2. $m \ne n$

If $i \le j$, then $\lambda_{i\,j} = (\lambda_i') + (\lambda_j'')$.

If $i > j$, then $\lambda_{2\mathrm{i}-j} = (\lambda_i') + (\lambda_j'')$.

If $i = m$ and $j = n$, then $\lambda_{2\mathrm{i}-j} = (\lambda_i') + (\lambda_j'')$.

5. Summary and Recommendations

To summarize, it was observed that whether one operates on two cycles, two paths, or a cycle and a path, the eigenvalue of the cross product of the two graphs was the product of the eigenvalues of each graph, while the eigenvalue of the box product of the two graphs was the sum of the eigenvalues of each graph.

In addition, iterative formulas or counters were obtained that could generate all the eigenvalues of the cross and box products in a certain order.

Since this paper is limited in terms of the number of vertices of paths and cycles, a possible study in the future would be to look at box and cross products of paths and cycles with higher order and verify if the same properties and iterative formulas constructed by the authors will still hold.

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