Domination, Independence and Other Numbers Associated With the Intersection Graph of a Set of Half-planes

Leonor Aquino-Ruivivar
Mathematics Department, De La Salle University
Leonor.ruivivar@dlsu.edu.ph

Abstract: Let l be a line on the Cartesian plane defined by an equation of the form $Ax + By + C = 0$. This induces two half-planes, namely:

$$H_r = \{ (x, y) \in \mathbb{R}^2 : Ax + By + C > 0 \}$$
$$H_l = \{ (x, y) \in \mathbb{R}^2 : Ax + By + C < 0 \}$$

called the right and left half-planes, respectively, induced by the line l. Let $\mathcal{L}_n$ be any set of n lines in the plane, and let $G(\mathcal{L}_n)$ be the intersection graph whose vertex set is the set of 2n half-planes induced by the lines in $\mathcal{L}_n$. Two half-planes are adjacent in this graph if they have a non-empty intersection.

The graph $G(\mathcal{L}_n)$ may be generalized by considering an arbitrary set $\mathcal{F}_n$ of n half-planes and denoting the corresponding intersection graph of these n half-planes by $\Omega(\mathcal{F}_n)$. In a previous study, some properties of these two intersection graphs were studied and identified, such as the diameter, the clique number, the maximum and minimum vertex degrees, the chromatic number and the vertex connectivity. Necessary and sufficient conditions for these graphs to belong to the classes of hamiltonian and eulerian graphs were also identified.

In the present study, certain graph invariants will be identified for the two intersection graphs. In particular, we will identify the domination and independence numbers, as well as numbers associated with various vertex, edge and total labelings of these graphs. A subset S of the vertex set V of a graph G is said to be *independent* if no two vertices in S are adjacent. The cardinality of a maximal independent subset of V is called the *independence number* of G. On the other hand, if x is a vertex of a graph G, then the *neighbor set* of x, denoted by $N(x)$, consists of all vertices adjacent to x. In this case, we say that *x dominates itself and each of its neighbors.* A subset D of the vertex set V of a graph G is said to be a *dominating set* of G if every vertex in G is either in D or is dominated by at least one element of D. The cardinality of a minimal dominating set for a graph G is called the *domination number* of G.

Key Words: intersection graph; vertex labelings; dominating set; domination number; independent set; independence number

1. INTRODUCTION

By a ***graph*** we mean a pair G = (V(G),E(G)) where V(G) is a nonempty set of elements called the *vertices* of G, and E(G) consists of unordered pairs called *edges* of elements of V(G). The numbers |V(G)| and |E(G)| are called the *order* and the *size* of the graph G, respectively.

With respect to certain graph properties, certain values or *graph invariants* may be assigned to a graph. In the present study, we will identify some of these graph invariants for the intersection graphs $G(\mathcal{L}_n)$ and $\Omega(\mathcal{F}_n)$.

To facilitate the determination of these invariants for these intersection graphs, we introduce the following notation. First, we partition the lines that serve as boundaries for the half-planes into disjoint classes of parallel lines, and denote these classes by $S_1, S_2, \ldots, S_k$. Let $|S_i| = s_i, i = 1,2,\ldots,k$, with $s_1 \leq s_2 \leq \cdots \leq s_k$. To the graphs $G(\mathcal{L}_n)$ and $\Omega(\mathcal{F}_n)$, we can associate the vectors

$$s(\mathcal{L}_n) = < s_1, \ldots, s_k >$$

$$s(\mathcal{F}_n) = < s_1, \ldots, s_k >$$

respectively. As in (Ruivivar, 2014), we assume that no two half-planes share a common boundary in $\Omega(\mathcal{F}_n)$.

2. DOMINATION NUMBER

Let G be a graph and let $v \in V(G)$. The *neighborhood (or open neighborhood)* $N(v)$ of v consists of all vertices of G which are adjacent to v. Thus,

$$N(v) = \{ u \in V(G) : [u, v] \in E(G) \}$$

We say that v *dominates* itself and each of its neighbors.

A subset S of V(G) is said to be a *dominating set* of G if every vertex in G is dominated by at least one vertex in S. A *minimum dominating set* of G is a dominating set with minimum cardinality, and this minimum cardinality is called the *domination number* of G, and is denoted by $\gamma(G)$.

Example 1
In Figure 1,, the set S consisting of the four shaded vertices is a minimum dominating set for the given graph G. Thus, $\gamma(G)$. =4.

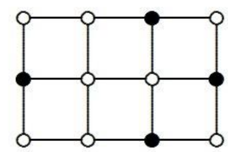

Figure 1: A graph with a minimum dominating set

The following theorem gives both upper and lower bounds for the domination number of a graph G. The notation $\Delta(G)$ represents the maximum degree of a vertex in G.

Theorem 1: Let G be a graph of order n. Then

$$\left\lfloor \frac{n}{1+\Delta(G)} \right\rfloor \leq \gamma(G) \leq n - \Delta(G)$$

In (Ruivivar, 2014), it was shown that $\Delta(G(\mathcal{L}_n) = 2n - 2$. Since the order of $G(\mathcal{L}_n)$ is 2n, Theorem 1 gives us the inequality

$$2 = \left\lfloor \frac{2n}{1+(2n-2)} \right\rfloor \leq \gamma(G) \leq 2n - (2n-2) = 2.$$

This gives us the following result:

Theorem 2: Let $n \geq 1$ be a positive integer. Then $\gamma(G(\mathcal{L}_n) = 2$.

For the domination number of the graph $\Omega(\mathcal{F}_n)$, we consider the following three cases:

Case 1: $(\mathcal{F}_n) = < n >$: This means that the boundaries of all the half planes are parallel. If $n \geq 2$ and there is at least one right half-plane whose boundary lies to the right of the boundary of at least one left half-plane, then $\gamma(\Omega(\mathcal{F}_n) = 2$. Otherwise, $\gamma(\Omega(\mathcal{F}_n) = 1$.

Case 2: $s(\mathcal{F}_n) = < 1,1,\ldots,1 >$: In this case, the boundaries of all the half-planes are pairwise intersecting, so that $\Omega(\mathcal{F}_n) \cong K_n$. Hence, $\gamma(\Omega(\mathcal{F}_n) = 1$.

Case 3: $s(\mathcal{F}_n) = < s_1, \ldots, s_k >, k \geq 2, s_k > 1$: In this case, the boundaries of the half-planes partition into k classes of parallel lines. Since any two half-planes whose boundaries belong to two different parallel classes are adjacent in $\Omega(\mathcal{F}_n)$, a dominating set must consist of half-planes whose boundaries belong to the same parallel class. From Case 1, $\gamma(\Omega(\mathcal{F}_n)$ is either 1 or 2.

3. INDEPENDENCE NUMBER

Let G be a graph and let $S \subseteq V(G)$ be non-empty. If the vertices in S are pairwise non-adjacent, we say that S is an *independent set* in G. The maximum cardinality of an independent set in G is called the *independence number* of G and is denoted by $\beta(G)$.

Example 2
In Figure 1 below, vertices x_1 and x_2 form an independent set. The set $S = \{x_1, x_2, x_5\}$ is an independent set of the given graph with maximum cardinality. Hence, $\beta(G) = 3$.

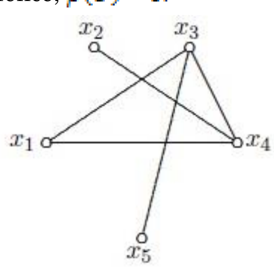

Figure 2: Independent set in a graph

For the graph $G(\mathcal{L}_n)$, we know that a pair of left and right half-planes which share a common boundary are non-adjacent in $G(\mathcal{L}_n)$, that half-planes with non-parallel boundaries are adjacent, and a set of all left (or all right) half-planes with parallel boundaries are adjacent. Hence, we have the following result:

Theorem 3: $\beta(G(\mathcal{L}_n) = 2$.

For the graph $\Omega(\mathcal{F}_n)$, we again consider the parallel classes of boundaries for the n half-planes. If the lines are pairwise intersecting, then $\Omega(\mathcal{F}_n) \cong K_n$ and $\beta(\Omega(\mathcal{F}_n)) = 1$. If the boundaries are all parallel lines, and there is at least one left half-plane whose boundary lies to the left of at least one right half-plane, then $\beta(\Omega(\mathcal{F}_n)) = 2$; otherwise, $\beta(\Omega(\mathcal{F}_n)) = 1$. Finally, if $s(\mathcal{F}_n) = < s_1, \ldots, s_k >, k \geq 2, s_k \geq 2$, let $\mathcal{F}_i$ be an equivalence class of half-planes with parallel boundaries such that $|\mathcal{F}_i| \geq 2$. If, as in the preceding case, there is at least one left half-plane whose boundary lies to the left of at least one right half-plane, then $\beta(\Omega(\mathcal{F}_n)) = 2$; otherwise, $\beta(\Omega(\mathcal{F}_n)) = 1$. Thus, we obtain the following result.

Theorem 4: Let $n \geq 2$. If there exist one left-half-plane and one right half-plane with parallel boundaries such that the boundary of the left half-plane lies to the left of the boundary of the right half-plane, then $\beta(\Omega(\mathcal{F}_n)) = 2$; otherwise, $\beta(\Omega(\mathcal{F}_n)) = 1$.

4. CHROMATIC NUMBER

Let G be a graph, let k be a positive integer and let $S = \{1, 2, \ldots, k\}$ be a set of *colors* or *labels*. A *proper vertex coloring* of G is a function $\varphi: V(G) \to S$ such that such that if vertices u and v are adjacent in G, then $\varphi(u) \neq \varphi(v)$. The graph G is said to be *k-colorable* if there exists a proper vertex coloring of G which uses a maximum of k colors, and the *chromatic number* of G, denoted by $\chi(G)$, is the smallest number k such that G is k-colorable.

Example 3:
The graph below is the wheel W_5.

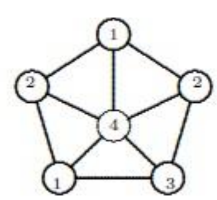

Figure 3: A 4-coloring of the wheel W_5

Since the central vertex is adjacent to each of the outer vertices, it must receive a color different from the colors used for the outer vertices. It can be easily verified that three colors are needed for the outer vertices. Thus, $\chi(W_5) = 4$.

For the graph $G(\mathcal{L}_n)$, the n left half-planes form a graph isomorphic to the complete graph K_n, so n colors are needed to properly color the vertices corresponding to the left half-planes. Since left and right half-planes which share a common boundary are non-adjacent in $G(\mathcal{L}_n)$, these half-planes can be assigned the same color. Thus, no additional color is needed to color the vertices corresponding to the right half-planes. This establishes the following:

Theorem 5: Let $n \geq 1$. Then $\chi\big(G(\mathcal{L}_n)\big) = n$.

For the graph $\Omega(\mathcal{F}_n)$ to have chromatic number n, it must be true that $\Omega(\mathcal{F}_n) \cong K_n$. This will happen if any of the following conditions are satisfied:

Proposition 1: The graph $\Omega(\mathcal{F}_n)$ is isomorphic to the complete graph of order n only if one of the following conditions hold:

(a) $s(\mathcal{F}_n) = < 1,1,\ldots,1 >$
(b) $s(\mathcal{F}_n) = < n >$, and the boundary of every left half-plane is to the right of the boundary of each right half-plane.
(c) $s(\mathcal{F}_n) =< s_1,\ldots,s_k >, k \geq 2, s_k \geq 2$, and each parallel class $\mathcal{F}_i$ induces a subgraph $\Omega(\mathcal{F}_i)$ of $\Omega(\mathcal{F}_n)$ which is isomorphic to the complete graph K_{s_i}

The graph $\Omega(\mathcal{F}_n)$ will have chromatic number less than n if the conditions given in (b) and (c) of Lemma 1 fail to hold. In the case of (b), this means that at least one left half-plane has its boundary to the left of the boundary of at least one right half-plane. For (c), at least one of the parallel classes of lines that serve as boundaries for the half-planes induces a subgraph $\Omega(\mathcal{F}_i)$ which is not isomorphic to K_{s_i}. The chromatic number will be at its minimum under the following conditions:

Proposition 2: The chromatic number of the graph $\Omega(\mathcal{F}_n)$ is at its minimum under the following conditions:

(a) $s(\mathcal{F}_n) = < n >$, and the boundary of every left half-plane is to the right of the boundary of each right half-plane. If there are l left half-planes and r right half-planes, then $\chi\big(\Omega(\mathcal{F}_n)\big) = \max\{(l,r\}$.
(b) $s(\mathcal{F}_n) =< s_1,\ldots,s_k >, k \geq 2, s_k \geq 2$, and in each subgraph $\Omega(\mathcal{F}_i)$, the boundary of every left half-plane is to the right of the boundary of each right half-plane.

5. CHROMATIC INDEX

In the preceding section, we considered a labeling of the vertices of a graph G. We now consider labelings of the edges of G. Given a set $S = \{1,2,\ldots,k\}$ where k is some positive integer, a *proper edge coloring* of a graph G is a function $\varphi: E(G) \to S$ such that whenever *e* and *f* are two edges of G which are incident to a common vertex, then $\varphi(e) \neq \varphi(f)$. A *k-edge coloring* of G is a proper edge coloring which uses at most k colors, and the *edge chromatic number* or *chromatic index* of G, denoted by $\chi_e(G)$, is the smallest positive integer k such that G is k-edge colorable.

If $\Delta(G)$ is the maximum degree of a vertex in G, then the following gives the possible values of the chromatic index of a graph G:

Theorem 6 (Vizing): If G is a finite simple graph, then either $\chi_e(G) = \Delta(G)$ or $\chi_e(G) = \Delta(G) + 1$.

It has been shown that most of the common classes of graphs have chromatic index $\chi_e(G) = \Delta(G)$. Such graphs are called *graphs of class 1.* Graphs that belong to the other category are called *graphs of class 2.*

It is known that complete graphs of even order are of class 1, For graphs of odd order, the following result gives a sufficient condition for a graph to be of class 2.

Theorem 7(Beineka and Wilson, 1973) : Let G be a graph of order n = 2s + 1, size m and maximum vertex degree $\Delta(G) = d$. If $m > sd$, then G is a graph of class 2.

For odd n, we have $\Delta(K_n) = d = n - 1$, and

$$m = \frac{n(n-1)}{2} = ns > (n-1)s = ds$$

so by Theorem 7, complete graphs of odd order are of class 2. Thus by Proposition 1, when $s(\mathcal{F}_n)$ satisfies any one of the conditions of this proposition, then the intersection graph $\Omega(\mathcal{F}_n)$ is of class 1 if n is even, and of class 2 if n is odd. A general formula for the chromatic index of $\Omega(\mathcal{F}_n)$ is difficult to obtain due to the arbitrary sequencing of left and right half-planes in $\Omega(\mathcal{F}_n)$.

6. CONCLUSION

In this study, some graph invariants were identified for the intersection graphs of half-planes. Future research may consider working on other graph invariants, and towards a general formula that will give the values of these invariants for these intersection graphs.

7. REFERENCES

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