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On Bisupermodular Games

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Abstract:
We discuss a special type of bicooperative game called bisupermodular games. This type of game is analogous to the concept of convex game in the classic cooperative game theory. The players(or participants) of a bicooperative game decide to join either S(the defender coalition) or T(the detractor coalition) or choose $N\backslash(S\cup T)$ (which is equivalent to the act of abstinence). In this paper, we discuss properties of the core and Weber set of a bisupermodular game. We also include an alternative way of studying bicooperative games by way of transforming it into a restricted (classical) cooperative game. From this, we show the analogy between the concept of bisupermodularity and convexity of classical cooperative games. Finally, we introduce the concept of dominance core of bisupermodular games and showed its relationship with the core.

Keywords: bicooperative games, bisupermodular games, Weber, core, dominance core

1. INTRODUCTION

Given a finite set of players and a real-valued worth function, a *transferable utility game* is defined as a cooperative game wherein payoffs are awarded to each coalition such that the worth of the empty coalition is zero. The worth per coalition can be interpreted as the maximal gain or minimal cost that the players in that coalition can get against the complementary coalition. As a motivation for defining a bicooperative game, consider the situation wherein a certain population of voters must "vote for" or "vote against" passing a bill. The members of this population may be viewed as "players of the game" so that they can be classified into two groups: the *defenders*(or those in favor) and the *detractors*(or those not in favor). It may happen though that some players are not convinced of the benefits of the bill. But at the same time, they also have no intention in objecting the bill, and thus, there may be some who can be described as *neutrals*. This leads us into the concept of *bicooperative game*.

Let $N = \{1,...,n\}$ and $3^N = \{(A,B) : A,B \subseteq N, A\cap B = \emptyset\}$. In this paper, we consider the partial order in 3^N given by

$$(A,B) \sqsubseteq (C,D) \Longleftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

Strict inclusion is denoted by $\sqsubset$ wherein $(A,B) \sqsubset (C,D)$ if and only if $A \subset C$ and $B \supset D$.

The set $(3^N, \sqsubseteq)$ is a partially ordered set with the following properties:

1. $(\emptyset, N)$ is the *first element*:

$$(\emptyset, N) \sqsubseteq (A,B) \text{ for all } (A,B) \in 3^N.$$

2. $(N, \emptyset)$ is the *last element*:

$$(A,B) \sqsubseteq (N,\emptyset) \text{ for all } (A,B) \in 3^N.$$

3. Every pair $\{(A,B),(C,D)\}$ of element of 3^N has a *join* given by

$$(A,B) \vee (C,D) = (A\cup C, B\cap D)$$

and a *meet* given by

$$(A,B) \wedge (C,D) = (A\cap C, B\cup D).$$

Two pairs (A,B) and (C,D) in 3^N are *comparable* if $(A,B) \sqsubseteq (C,D)$ or $(C,D) \sqsubseteq (A,B)$. Otherwise, (A,B) and (C,D) are *non-comparable*.

Throughout this paper, we use $S\cup i$ and $S\backslash i$ instead of $S\cup\{i\}$ and $S\backslash\{i\}$, respectively. The number of players in S is denoted by $|S|$.

2. PRELIMINARY CONCEPTS

The *bicooperative game* is defined as follows.

Definition 1. *A bicooperative game is a pair* (N, b) *where* N *is a finite set of players and* $b : 3^N \to \mathbb{R}$ *is a function such that* $b(\emptyset, \emptyset) = 0$.

Given a player set N, we use the notation $\mathcal{BG}^N$ as the collection of all bicooperative games defined on N. The set N is partitioned in three coalitions: the set of players who participate or the defenders (denoted S), the set of players who act against or the detractors (denoted by T) and the set of players who abstain (denoted by $N \backslash S \cup T$).

In a bicooperative game, the amount $b(\emptyset, N)$ is the expense or cost whenever all of the players detracts and $b(N, \emptyset)$ is the gain acquired when all of the players are in favor or agreed. Therefore, the quantity $b(N, \emptyset) - b(\emptyset, N)$ can be viewed as the net profit. A solution concept for bicooperative games is a function used in assigning payoff vectors to every bicooperative game that distribute the net profit among all players. We introduce two solution concepts for bicooperative games in this section: the core and the Weber set.

A vector x in $\mathbb{R}^n$ which satisfies $\sum_{i \in N} x_i = b(N, \emptyset) - b(\emptyset, N)$ is an *efficient vector*. The set of all efficient vectors is the *preimputation set* defined as

$$I^*(N, b) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = b(N, \emptyset) - b(\emptyset, N)\}.$$

We see that if $x \in I^*(N, b)$ then it gives a partitioning of the whole net profit to all players in N.

An allocation wherein a player receives not less than what he can get without cooperating with other players is said to satisfy *individual rationality*. The *imputations for a game* (N, b) are the preimputations that satisfy individual rationality for all players. Thus, the imputation set that corresponds to b with player set N is given by

$$I(N, b) = \left\{ \begin{array}{r} x \in I^*(N, b) : x_i \geq b(i, N \backslash i) - b(\emptyset, N) \\ \text{for all } i \in N \end{array} \right\}.$$

3. SOME SOLUTION CONCEPTS OF BICOOPERATIVE GAMES

In the following discussions, we consider a payoff vector from a set called core of the game.

Definition 2. *Let* $b \in \mathcal{BG}^N$. *The core of* b *is the set*

$$C(N, b) = \left\{ \begin{array}{r} x \in I^*(N, b) : \textit{there exist } y, z \in \mathbb{R}^n \\ \textit{such that } x = y + z, \textit{ and} \\ y(S) + z(N \backslash T) \geq b(S, T) - b(\emptyset, N), \\ \textit{for all } (S, T) \in 3^N \end{array} \right\}$$

An element $x = (x_1, x_2, \ldots, x_n) \in C(N, b)$ represents a payoff vector of all the players N such that x_i is the payoff of Player i. Given $(S, T) \in 3^N$, players who do not belong to T receive payoffs determined by the vector $z \in \mathbb{R}^n$. But those players in S also get additional payoffs as determined by the vector $y \in \mathbb{R}^n$.

We skip the proof of the following proposition so that the readers are referred to [1] for details. We now discuss some properties of $C(N, b)$.

Proposition 1. $C(N, b)$ *is a bounded set.*

Proposition 2. *Let* $x \in I^*(N, b)$ *be such that* $x = y + z$. *Then for any* $(S, T) \in 3^N$,

$$\begin{gathered} y(S) + z(N \backslash T) \geq b(S, T) - b(\emptyset, N) \\ \textit{if and only if} \\ y(N \backslash S) + z(T) \leq b(N, \emptyset) - b(S, T). \end{gathered}$$

Proposition 3. *For every* $b \in \mathcal{BG}^N$, *its core* $C(N,\ b)$ *is a convex set.*

Here is another solution concept for cooperative games. Each permutation $\pi = (i_1, \ldots, i_n)$ of the elements of N in a classic cooperative game (N, v) represents a sequential process of the formation of the grand coalition N. The corresponding *marginal worth vector* $a^\pi \in \mathbb{R}^n$, gives the marginal contribution of every player to the coalition formed by his predecessors, that is $a^\pi_{i_j}(v) = v(\pi(i_j)) - v(\pi(i_j) \backslash \{i_j\})$ for all $i_j \in N$ where $\pi(i_j) = \{i_1, \ldots, i_j\}$ is the set of the predecessors of player i_j in the order π. The convex hull of all marginal worth vectors is called the Weber set of the game.

To extend the idea of the Weber set in the bicooperative game, we assume that all players estimate that $(N, \emptyset)$ is formed by a sequential process wherein each step a player enters the defender coalition or leaves the detractor coalition. This sequential process can be represented by a chain from $(\emptyset, N)$ to $(N, \emptyset)$. For each chain, a player can evaluate his contribution given as vectors in $\mathbb{R}^n$ when he joins the defenders or when he leaves the detractors.

These contributions are called *superior marginal worth vectors* and *inferior marginal worth vectors*. To formalize the idea, we introduce the following notations.

For $N = \{1,\ldots,n\}$, let $\overline{N} = \{-n,\ldots,-1,1,\ldots,n\}$. Let Λ: $3^N \to 2^{\overline{N}}$ be the isomorphism defined by $\Lambda(S,T) = S \cup \{-i : i \in N\backslash T\} \in 2^{\overline{N}}$, for each $(S,T) \in 3^N$. For instance, $\Lambda(\emptyset,N) = \emptyset$ and $\Lambda(N,\emptyset) = \overline{N}$. As $S \subseteq N\backslash T$, we see that $i \in \Lambda(S,T)$ and $i > 0$ imply $-i \in \Lambda(S,T)$

We identify a maximal chain to be

$$(\emptyset,N) \sqsubset (S_1,T_1) \sqsubset \cdots \sqsubset (S_j,T_j) \sqsubset \cdots$$

$$\sqsubset (S_{2n-1},T_{2n-1}) \sqsubset (N,\emptyset),$$

with an ordering $\theta = (i_1,\ldots,i_{2n})$ on $\overline{N}$ such that $\Lambda(S_j,T_j) = \theta(i_j)$ for all $j = 1,\ldots,2n$, where $\theta(i_j) = \{i_1,\ldots,i_j\}$ is the set of predecessors of i_j in the order θ and its elements are written following the order of incorporation in the defender coalitions or desertion from the detractor coalitions. If $i_j > 0, i_j$ is the last player who joins $S_j(i_j \in S_j$ and $i_j \notin S_{j-1})$ and, if $i_j < 0, -i_j$ is the last player who leaves $T_{j-1}(-i_j \notin T_j$ and $-i_j \in T_{j-1})$. In particular, $\Lambda^{-1}[\theta(i_{2n})] = (N,\emptyset)$ and $\Lambda^{-1}[\theta(i_1)\backslash i_1] = (\emptyset,N)$.

In $(3^N,\sqsubseteq)$, let $\Theta(3^N)$ denote the set of all maximal chains going from $(\emptyset,N)$ to $(N,\emptyset)$.

Definition 3. *Let $\theta \in \Theta(3^N)$ and $b \in \mathcal{BG}^N$. The inferior and superior marginal worth vectors with respect to θ are the vectors $m^\theta(b), M^\theta(b) \in \mathbb{R}^n$ given by*

$$m_i^\theta(b) = b(\Lambda^{-1}[\theta(-i)]) - b(\Lambda^{-1}[\theta(-i)\backslash -i]), \qquad (1)$$

$$M_i^\theta(b) = b(\Lambda^{-1}[\theta(i)]) - b(\Lambda^{-1}[\theta(i)\backslash i]), \qquad (2)$$

for all $i \in N$. The vector $a^\theta(b) = m^\theta(b) + M^\theta(b)$ is called the marginal worth vector with respect to θ.

Proposition 4. *Let $b \in \mathcal{BG}^N$ and $\theta \in \Theta(3^N)$. Then,*

$$\sum_{j\in S} M_j^\theta(b) + \sum_{j\in N\backslash T} m_j^\theta(b) = b(S,T) - b(\emptyset,N),$$

for every (S,T) in the chain θ.

Definition 4. *Let $b \in \mathcal{BG}^N$. The Weber set of b is the convex hull of the marginal worth vectors of b, that is, $W(N,b) = conv\{a^\theta(b) : \theta \in \Theta(3^N)\}$.*

Bilbao, Fernandez, Jimenez and Lopez [1] proved that in bicooperative games, the core is always included in its Weber set. Hence, the following theorem holds.

Theorem 1. *If $b \in \mathcal{BG}^N$, then*

$$C(N,b) \subseteq W(N,B).$$

4. BISUPERMODULAR GAMES

One special type of cooperative game is the convex game. It is a game v defined on the player set N satisfying the property that for any $S,T \subseteq N$,

$$v(S\cup T) + v(S\cap T) \geq v(S) + v(T).$$

If there is a convex game in the classical cooperative game, then what will be its counterpart in bicooperative game? This leads us to a special class of bicooperative game called bisupermodular games.

Definition 5. *A bicooperative game $b \in \mathcal{BG}^N$ is bisupermodular if, for all $(S_1,T_1),(S_2,T_2) \in 3^N$ we have*

$$b((S_1,T_1)\vee(S_2,T_2)) + b((S_1,T_1)\wedge(S_2,T_2))$$

$$\geq b(S_1,T_1) + b(S_2,T_2),$$

or equivalently

$$b(S_1\cup S_2, T_1\cap T_2) + b(S_1\cap S_2, T_1\cup T2)$$

$$\geq b(S_1,T_1) + b(S_2,T_2)$$

The next proposition discusses a characterization of bisupermodular games in terms of comparable members of 3^N and the contribution of a single player in joining the defender coalition or leaving the detractor coalition.

Proposition 5. *The bicooperative game b is bisupermodular if and only if for all $i \in N$ and for all $(S_1,T_1),(S_2,T_2) \in 3^{N\backslash i}$ such that $(S_1,T_1) \sqsubseteq (S_2,T_2)$, we have*

$$b(S_2\cup i,T_2) - b(S_2,T_2) \geq b(S_1\cup i,T_1) -b(S_1,T_1), \qquad (3)$$

$$b(S_2,T_2) - b(S_2,T_2\cup i) \geq b(S_1,T_1) -b(S_1,T_1\cup i). \qquad (4)$$

In the next theorem, it shows that the marginal worth vectors are in the core for every bisupermodular game.

Theorem 2. *A bicooperative game $b \in \mathcal{BG}^N$ is bisupermodular if and only if all the marginal worth vectors of b are in the core $C(N,b)$.*

The core is convex according to Theorem 3 and by the supplement of Theorem 2, the result of these two theorems is the following characterization.

Corollary 1. *A bicooperative game $b \in \mathcal{BG}^N$ is bisupermodular if and only if W(N,b) = C(N,b).*

5. RESTRICTED GAME

We can view bicooperative game (N, b) in another way [2]. Consider an alternative set of players defined by

$$\mathcal{N} = \{(i, t) : i \in N, t \in \{1, 2\}\}.$$

A coalition $K \subseteq \mathcal{N}$ is *feasible* if $(i, 2) \in K$ implies $(i, 1) \in K$. The set of feasible coalitions is denoted by $\mathcal{F} \subseteq \mathcal{N}$.

One may think of the feasibility of a coalition $K \subseteq \mathcal{N}$ as follows:

- K forms two sets
$$K_1 = \{i : (i, 1) \in K\},$$
$$K_2 = \{i : (i, 2) \in K\}.$$
- Associate K_2 with the defenders and K_1 with the non-detractors.
- The condition $(i, 2) \Rightarrow (i, 1) \in K$ simply says that if a player is a defender then he is a non-detractor.

With this argument, we can now associate every $K \in \mathcal{F}$ with a member $(S, T) \in 3^N$ so that $S = K_2$ and $T = (N \backslash K_1)$.

Theorem 3.

$$(\mathcal{F}, \cup, \cap) \cong (3^N, \vee, \wedge)$$

with the mapping $\Phi : (\mathcal{F}, \cup, \cap) \to (3^N, \vee, \wedge)$ *given by* $\Phi(K) = (K_2, N \backslash K_1)$ *where* $K_t = \{i \in N : (i, t) \in K\}, t \in \{1, 2\}$.

We take note that if $\Phi(K) = (K_2, N \backslash K_1) = (S, T)$, then $\Phi^{-1}(S, T) = K = \{(i, 2) : i \in K_2\} \cup \{(i, 1) : i \in N \backslash K_1\}$. Thus, the inverse mapping $\Phi^{-1}(S, T) \cong \Lambda(S, T)$ since $\Lambda(S, T) = S \cup (-i : i \in N \backslash T)$. Actually, $\Phi^{-1}(S, T)$ lists the same set of players that are described in $\Lambda(S, T)$.

Using the isomorphism above, given $b \in \mathcal{BG}^N$, we define the game $v : \mathcal{F} \to \mathbb{R}$ as

$$v(K) = b(K_2, N \backslash K_1) - b(\emptyset, N). \quad (5)$$

Also note that $\Phi(\emptyset) = (\emptyset, N)$. Now, we define the restricted game $(\mathcal{F}, v)$ where $\mathcal{F} \subseteq \mathcal{N}$ is the set of feasible coalition and v is the value of the game.

Theorem 4. *The bicooperative game (N, b) is bisupermodular if and only if the restricted game $(\mathcal{F}, v)$ is a convex game.*

6. DOMINANCE CORE IN BICOOPERATIVE GAME

Before we define *dominance core*, we first define the concept of x *dominates* y wherein x and y is in the imputation set.

Definition 6. *Let $x, y \in I(N, b)$. We say that* x dominates y *if there exists $(S, T) \in 3^N$ where $(S, T) \neq (\emptyset, N)$ such that*

(i) $x_i > y_i$ for all $i \in S \cup N \backslash T$, and

(ii) $\sum_{i \in S \cup N \backslash T} x_i \leq b(S, N \backslash T) - b(\emptyset, N)$.

We write $x \; dom_{(S,T)} \; y$ to mean that x dominates y with respect to (S, T). Also, we say that x is a *dominating imputation*. We observe that the players who are not detractors get better payoff in x than in y. Moreover, if the detractors insist on adapting y in the computation of their payoffs, the other members can openly reject this idea because property (ii) of Definition 6 shows efficiency in $x(S \cup N \backslash T)$.

Definition 7. *The set $DS = \{y \in I(N, b) | \; x \; dom_{(S,T)} \; y$ for some $x \in I(N, b)$ and some $(S, T) \in 3^N \backslash \{(\emptyset, N)\}\}$ is called the* dominated set *of the bicooperative game (N, b).*

We define the *dominance core* of the bicooperative game (N, b) as

$$DC(N, b) = I(N, b) \backslash DS.$$

Based on the above description of $DC(N, b)$, we see that an imputation $x \in DC(N, b)$ if there is no imputation y such that $y \; dom_{(S,T)} \; x$ for all $(S, T) \neq (\emptyset, N)$.

Theorem 5. *If $b \in \mathcal{BG}^N$, then*

$$C(N, b) \subseteq DC(N, b).$$

Proof. Assume $x \in C(N, b)$ but $x \notin DC(N, b)$. Then there exists $y \in I(N, b)$ and $(\emptyset, N) \neq (S, T)$ such that

$$y \; dom_{(S,T)} \; x$$

Hence,

$$y_i > x_i \text{ for all } i \in S \cup N \backslash T \quad (6)$$

and

$$\sum_{i \in S \cup N \backslash T} y_i \leq b(S, T) - b(\emptyset, N). \quad (7)$$

Now, since $x \in C(N,b)$, there exist $w, z \in \mathbb{R}^n$ such that $x = w + z$ and in particular

$$\begin{aligned} x(S \cup N\backslash T) &= w(S) + z(N\backslash T) \\ &\geq b(S,T) - b(\emptyset, N). \end{aligned} \tag{8}$$

But (6) implies

$$\sum_{i \in S \cup N\backslash T} y_i > \sum_{i \in S \cup N\backslash T} x_i. \tag{9}$$

From inequalities (7), (8) and (9),

$$\begin{aligned} b(S,T) - b(\emptyset,N) \leq x(S \cup N\backslash T) &= \textstyle\sum_{i \in S\cup N\backslash T} x_i \\ < \textstyle\sum_{i\in S\cup N\backslash T} y_i &\leq b(S,T) - b(\emptyset, N) \end{aligned}$$

which is a contradiction. □

7. SUMMARY AND CONCLUSION

The analogy between classical games and bicooperative games are shown below.

The theory of bicooperative game studies different strategies of the players by forming coalitions in order to receive the best payoff. The players can agree or disagree with the proposal or they can even stay neutral. The core in bicooperative games shows that the players who do not act against the coalition will receive payoffs greater than what they can receive if they act against the rest of the players. The players who joined the defender coalition will get more than those who stayed neutral. Moreover, the Weber set is the convex hull of the marginal worth vectors of the payoffs.

This study has shown that the core is a subset of the Weber set in bicooperative games. However, they are equal in bisupermodular games. This paper is able to show an alternative way of viewing any bicooperative game. Furthermore, this paper shows the transformation of classical sense of convexity into bisupermodular games. Finally, the solution concept of dominance core is viewed in bicooperative games and it shows that the core is a subset of dominance core in bicooperative games.

	Classical	Bicooperative
Player set	N	N
Characteristic function	$v : 2^N \to \mathbb{R}$ $2^N = \{S : S \subseteq N\}$ where $v(\emptyset) = 0$	$b : 3^N \to \mathbb{R}$ $3^N = \{(S,T) : S, T \subseteq N,$ $S \cap T = \emptyset\}$ where $b(\emptyset,\emptyset) = 0$
Core	$\sum x_i \geq v(S)$	$x = y + z$ and $y(S) + z(N\backslash T)$ $\geq b(S,T) - b(\emptyset, N)$
Weber	$conv\{a^\pi(v) \in \mathbb{R}^n :$ $\pi \in \prod_N\}$	$conv\{a^\theta(b) : \theta \in \Theta(3^N)\}$
Dominance Core	$x \text{ dom}_S y$ $x_i > y_i$ for all $i \in S$ $\sum_{i\in S} x_i \leq v(S)$	$x \text{ dom}_{(S,T)} y$ $x_i > y_i$ for all $i \in S \cup N\backslash T$ $\sum_{i \in S\cup N\backslash T} x_i$ $\leq b(S, N\backslash T) - b(\emptyset, N)$
Convex/ Bisupermodular	$v(S\cup T) + v(S \cap T)$ $\geq v(S) + v(T)$ where $S, T \subseteq N$	$b((S_1,T_1) \vee (S_2,T_2))$ $+\ b((S_1,T_1) \wedge (S_2,T_2))$ $\geq b(S_1,T_1) + b(S_2,T_2)$ where $(S_1,T_1),(S_2,T_2) \in 3^N$
	$(\mathcal{F}, v)$	(N, b)

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