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On the Core and Star Coalitions of Bicooperative Games

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Abstract:
In this paper, we focus on a certain variation of the classic cooperative game called bicooperative game. In this type of game, the players (or participants) are asked to make a choice to vote yes, to vote no, or to abstain. Thus, the player set N is partitioned into three sets S, T, and $N \backslash (S \cup T)$. The question now lies on how the members of N receive their payoffs based on the outcome of the game. This leads us to the concept of solutions to bicooperative games. Assuming that the players want to maximize their own rewards, allocating these rewards in a fair manner to all players of each coalition is a problem that interests those who study game theory.

This study aims to give a discussion on some solution concepts of bicooperative games. The preliminary concepts are taken from the paper entitled The core and the Weber set for bicooperative games by Bilbao, Fernandez, Jimenez and Lopez.

Now, there are situations in which a few players are considered 'superior' or 'VIP' due to their influence and power, especially when they are all in the same coalition. Each of these players can be persuaded to join a coalition in exchange for a higher payoff for them. These situations inspired the authors to introduce the concept of star coalitions (including the existence of a dictator) which is analogous to the notion of clan games from the classic cooperative game theory.

Keywords: bicooperative games; allocations; Core, star coalitions

1. INTRODUCTION

We are constantly presented with a lot of of choices that require decisions. Each decision will either bring goodness or destruction and this choice may also affect others. Correct choices must be made to obtain an effect that does not involve too many negative effects - an optimal outcome. Inspired by real life situations, games are invented that parallel real life decision making.

In the classic cooperative game theory, we define a cooperative game to be the ordered pair $\langle N, v \rangle$ where N is the set of all players and v is a characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The function assigns to every possible subset of N a real number. This real number may be thought of as the total payoff of a coalition which is to be divided among all of its members.

The solution concept of a classic cooperative game is the allocation of the total payoff once the grand coalition N is formed. Some examples of solution concepts are Shapley value, core, dominance core and Weber set. The core in a cooperative game theory is a set of payoff vectors such that all coalitions have a feasible payoff and no coalition is put under in an unfavorable position (less payoff than what they can attain from the coalition itself).

In this paper, we focus on a certain variation of a game called *bicooperative game*. Bicooperative games ask players (or participants) to make a choice among cooperation, competition or abstinence. Every player would like to obtain the best choice in order to attain an acceptable or optimal reward. Of course, choosing the option that offers the highest reward should be the most desirable.

Players cooperate with each other to form teams (coalitions) in search of greater rewards than what they can obtain by themselves. Assuming that the players want to maximize their own rewards, allocating these rewards in a *'fair'* manner to all players of each coalition is a problem that interests those who study game theory. The allocation of rewards are what we call the solutions of bicooperative games. The solutions that we define impose some properties to be satisfied by the allocation.

In some games, a few players are considered 'superior' or 'VIP' due to their influence and power, especially when they are all in the same coalition. Each of these players can be persuaded to join a coalition in exchange for more payoff for them.

This study primarily aims to give a detailed discussion of the paper on the solutions of bicooperative games by Bilbao, Fernandez, Jimenez and Lopez [2]. Specifically, it aims to present solution concepts of bicooperative games and to relate these to core and Weber sets and its applications.

Since the proposition of bicooperative games in 2000, there have only been a few studies regarding this subject. A bicooperative game is different from a cooperative game in the sense that in a cooperative game its focus is the formulation of coalitions resulting in the partitioning of the player set N into two sets which are S and $N \backslash S$. On the other side, bicooperative games focuses on the formulation of two subsets of N and thus, resulting in the partitioning of the player set N into 3 sets: S (Yes votes), T (No votes) and $N \backslash S \cup T$ (abstain). Another difference between cooperative and bicooperative game is the computation of net payoffs where this value will be the one to be allocated among the members of N. In a cooperative game its net payoff is computed by $v(N)$ however in a bicooperative game the net payoff is computed by subtracting the value of the game when all of the players says 'No' from the value of the game when all of the players says 'Yes'.

In this paper, we study the solution concepts of bicooperative games called core. Moreover, the authors include extensions of the concept of clan games applied to bicooperative game as original contributions. The study limits its scope to bicooperative games only.

2. BICOOPERATIVE GAMES

One of the main motivations for the formation of the concept of bicooperative games is the example of ternary voting games. In ternary voting, voters who are in the grand coalition N are presented a proposition and they have the option to accept, reject, or abstain. Abstaining means that the player is not convinced of the benefits of the proposition, but is neither against it. In this case, the set S consists of the voters that accepted the proposition or who voted 'yes' and the set T contains the ones who rejected it or who voted 'no'. We also call the coalition S as the *defender coalition* while the coalition T is called *detractor coalition*.

Since the players have three choices: Accept(Yes), Reject(No), or abstain, and there are N players in the game, then there are $3^{|N|}$ possible outcomes of the game. The notation (S, T) represents the sets that contain players that behave in a positive and a negative way, respectively. For this definition, sets S and T are assumed to be disjoint.

Definition 1. *A* ***bicooperative game*** *is a pair* (N, b) *where* N *is a finite set and* b *is the function* $b : 3^N \to \mathbb{R}$ *with* $b(\emptyset, \emptyset) = 0$.

Example 1 (Investment Problem)**.** *S.T.A.R. company have 3 investors namely Alice, Bob, and Charlie. The investments held by the company amount to 3,5, and 7 (in million dollars) respectively. The S.T.A.R. company is now proposing a project wherein it is expected to receive 5 million for every 1 million investment. Now, S.T.A.R. company would like to ask the investors if they are willing to put their investment to the said project.*

Assuming the forecast of receiving 5M per 1M, a bicooperative game can be formed where saying 'yes' means that the investor agreed to put his/her investment to the said project. This means that if the investor says 'yes' the company would have a profit of 4M for every 1M that he invested. By saying 'no' the investor does not agree to the project and he would like to withdraw his investment to the company. Therefore, the company will have a loss of what the investor invested. Lastly, abstaining means the investor would not invest his/her project to the said project but would not increase or gain any profit.

We assign A for Alice, B for Bob and C for Charlie so that the player set is $N = \{A, B, C\}$. *We can compute the payoff when both Alice and Bob agree to invest in the project and Charlie declines and withdraws his investment.*

$$b(AB, C) = \underbrace{3}_{Alice's}(4) + \underbrace{5}_{Bob's}(4) - \underbrace{7}_{Charlie's} = 25$$

Next, suppose we want to compute for the payoff when Alice agrees to invest in the project, Bob abstains, while Charlie declines and withdraws his investment, then

$$b(A, C) = \underbrace{3}_{Alice's}(4) + \underbrace{0}_{Bob's}(4) - \underbrace{7}_{Charlie's} = 5$$

We compute the payoff for every (S, T) *in the same manner. The payoffs are enumerated on the table below.*

3. THE CORE

We will use the payoffs given by the first and last elements of the partially ordered set as defined, namely $b(\emptyset, N)$ and $b(N, \emptyset)$, to define the net income of the game. Firstly, $b(\emptyset, N)$ is interpreted as the expense or cost when all players oppose a proposition. On the other hand, $b(N, \emptyset)$ is interpreted as the income or gain when the proposed plan is agreed upon. Therefore, the net income is computed as $b(N, \emptyset) - b(\emptyset, N)$.

$(S,T) \in 3^N$	$b(S,T)$	$(S,T) \in 3^N$	$b(S,T)$	$(S,T) \in 3^N$	$b(S,T)$
$(N,\emptyset)$	60	(A,B)	7	$(AB,\emptyset)$	32
$(A,\emptyset)$	12	(A,C)	5	$(AC,\emptyset)$	40
$(B,\emptyset)$	20	(B,C)	13	$(BC,\emptyset)$	48
$(C,\emptyset)$	28	(B,A)	17	$(\emptyset,AB)$	-8
$(\emptyset,N)$	-15	(C,A)	25	$(\emptyset,AC)$	-10
$(\emptyset,A)$	-3	(C,B)	23	$(\emptyset,BC)$	-12
$(\emptyset,B)$	-5	(AB,C)	25	(C,AB)	20
$(\emptyset,C)$	-7	(AC,B)	35	(B,AC)	10
$(\emptyset,\emptyset)$	0	(BC,A)	45	(A,BC)	0

Table I: S.T.A.R. Company's Possible Profits

3.1. Preimputation and Imputation Set

Definition 2. *The* ***preimputation*** *set is the set of vectors $x \in \mathbb{R}^n$ satisfying the condition that*

$$I^*(N,b) = \left\{ x \in \mathbb{R}^n : \sum_{i\in N} x_i = b(N,\emptyset) - b(\emptyset,N) \right\}$$

Definition 3. *The vectors x in the preimputation set satisfying the individual rationality principle (i.e. every player must get no less than what he can gain by himself) are the elements of the* ***imputation set*** *given by*

$$I(N,b) = \{x \in I^*(N,b) : x_i \geq b(i,N\backslash i) - b(\emptyset,N)\}$$

Example 2. *From Example **??**, we can compute its preimputation and imputation set as follows:*

Define $x \in \mathbb{R}^3$ such that $x = (x_A, x_B, x_C)$.
With $b(N,\emptyset) - b(\emptyset,N) = 60 - (-15) = 75$, the preimputation set is:

$$I^*(N,b) = \{(x_A,x_B,x_C) | x_A + x_B + x_C = 75\}.$$

In order to determine the imputation set we consider the condition $x_i \geq b(i,N\backslash i) - b(\emptyset,N)$ for every $i \in N$. Thus,

$$b(A,BC) - b(\emptyset,N) = 15 \leq x_A,$$

$$b(B,AC) - b(\emptyset,N) = 25 \leq x_B,$$

$$b(C,AB) - b(\emptyset,N) = 35 \leq x_C.$$

Since x is in the preimputation set, it must satisfy the property that $\sum_{i\in N} x_i = 75$ The only $x \in \mathbb{R}^N$ that can satisfy the properties presented above is $x = (x_A,x_B,x_C) = (15,25,35)$. It can be verified that any addition to any x_i will violate the property of preimputation. Thus,

$$I(N,b) = \{(15,25,35)\}.$$

3.2. The Core

Every pair $(S,T) \in 3^N$ must at least receive an amount that it contributes to $(\emptyset,N)$ which we can denote as $b(S,T) - b(\emptyset,N)$. We also consider two sets of players that contribute to the formation of each $(S,T) \in 3^N$ that will lead us to the definition of the core of a bicooperative game. The two sets are S (defenders) and $N\backslash T$ (non-detractors). Payoffs of the members of these sets are determined by values $y,z \in \mathbb{R}^n$ so that since $S \subseteq N\backslash T$, members of S are rewarded 'twice'.

Definition 4. *The* ***core*** *is defined to be*
$C(N,b) = \{x \in I^*(N,b) : \exists y,z \in \mathbb{R}^n$ such that $x = y + z$, and $y(S) + z(N\backslash T) \geq b(S,T) - b(\emptyset,N), \forall (S,T) \in 3^N\}$.

Let $x \in C(N,b)$ and $i \in N$. Since $x_i = y_i + z_i \geq b(i,N\backslash i) - b(\emptyset,N)$, we obtain $x \in I(N,b)$. This also means that every element x in the core is also an imputation.

Example 3 (Core and its elements)**.** *Consider the player set $N = \{1,2\}$ and the function b determined by Table **??**.*

$(S,T) \in 3^N$	$b(S,T)$	$(S,T) \in 3^N$	$b(S,T)$
$(\emptyset,N)$	8	$(\emptyset,1)$	5
$(1,\emptyset)$	10	$((\emptyset,2)$	7
$(2,\emptyset)$	9	$(1,2)$	11
$(N,\emptyset)$	15	$(2,1)$	9

Table II: Payoff Table

Define $x = (x_1,x_2) \in \mathbb{R}^2$. The elements in the core are preimputations that satisfy the individual rationality principle. The elements of the core lie on the line $x_1 + x_2 = 7 (= b(N,\emptyset) - b(\emptyset,N)$.

In order to satisfy the individual rationality condition, we consider all possible $(S, T) \in 3^N$ yielding the following list of inequalities.

$b(S,T) \in 3^N$	$y(S) + z(N \setminus T) \geq b(S,T) - b(\emptyset, N)$
$(1, \emptyset)$	$y(1) + z(N) \geq 10 - 8 = 2 \Rightarrow x_1 \geq 2$
$(2, \emptyset)$	$y(2) + z(N) \geq 9 - 8 = 1 \Rightarrow x_2 \geq 1$
$(\emptyset, 1)$	$y(\emptyset) + z(2) \geq 5 - 8 = -3 \Rightarrow x_2 \geq -3$
$(\emptyset, 2)$	$y(\emptyset) + z(1) \geq 7 - 8 = -1 \Rightarrow x_2 \geq -1$
$(1, 2)$	$y(1) + z(1) \geq 11 - 8 = 3 \Rightarrow x_1 \geq 3$
$(2, 1)$	$y(2) + z(2) \geq 9 - 8 = 1 \Rightarrow x_2 \geq 1$

Table III: Computation of the core

Combining all these inequalities, we conclude that $x_1 \geq 3$ and $x_2 \geq 1$.

Thus, the core is the segment defined by

$$\{(x_1, x_2) | x_1 + x_2 = 7, 3 \leq x_1 \leq 6\}.$$

4. STAR COALITIONS AND DICTATOR

It is possible that there exists a group of influential players forming a group of star players. In this case, the members of the group can only achieve its influential position if they are strictly together in one coalition.

4.1. Clan Games

The motivation of the extension is the notion of clan games in the classic cooperative game. Its definition and properties are shown below.

Definition 5. *A game $v \in G^N$ is a* ***clan game*** *with clan $C \in 2^N \setminus \{\emptyset, N\}$ if it satisfies the following four conditions:*

(a) Nonnegativity: $v(S) \geq 0$ for all $S \subset N$;

(b) Nonnegative marginal contributions to the grand coalition: $M_i(N, v) \geq 0$ for each player $i \in N$;

(c) Clan property: every player $i \in C$ is a veto player, i.e. $v(S) = 0$ for each coalition S that does not contain C;

(d) Union property: $v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(N, v)$ if $C \subset S$.

Some theorems in clan games are the following:

Theorem 1. *Let $v \in G^N$ be a clan game. Then*

$$C(v) = \{x \in I(v) | x_i \leq M_i(N, v) \text{ for all } i \in N \setminus C\}.$$

Theorem 2. *Let $v \in G^N$ and $v \geq 0$. The game v is a clan game iff*

(i) $v(N)e^j \in C(v)$ for all $j \in C$;

(ii) There is at least one element $x \in C(v)$ such that $x_i = M_i(N, v)$ for all $i \in N \setminus C$.

4.2. Star Coalitions

With the definition of a clan game in the classical sense, the definition of a star coalition can be introduced.

In this section we assume each of the following:

(1) $b(S, T) \geq b(\emptyset, N)$ for all $(S, T) \in 3^N$;

(2) $M_i^d(N, b) = b(N, \emptyset) - b(N \setminus i, i) \geq 0 \quad \forall i \in N$

Definition 6. *Let $S^\star = \{p_1, \ldots, p_k\} \subset N$. We call $S^\star$ a* ***star coalition*** *if*

(i) for all $(S, T) \in 3^N$ with $p_m \notin S$ for some $m \in \{1, \ldots, k\}$ we have

$$b(S, T) - b(\emptyset, N) = 0; \quad \text{and}$$

(ii) Whenever $S^\star \subseteq N \setminus T$,

$$\sum_{i \in T} M_i^d(N, b) \leq b(N, \emptyset) - b(S, T).$$

Example 4. *From Example **??**, we now assume that A and B are members of the star coalition. That is, $\{A, B\} = S^\star$.*

Suppose we want to compute the payoff when only Alice agrees to invest in the project and both Bob and Charlie decline and withdraw their investment. then

$$b(AC, B) = \underbrace{3}_{Alice's}(4) - \underbrace{5}_{Bob's} + \underbrace{7}_{Charlie's}(4) = 35$$

The payoff must have been 35 but since Alice and Bob are not both in S, then the payoff will be equal to $b(\emptyset, N) = -15$ Take note that all the members of the star coalition must be in S to gain a payoff greater than $b(\emptyset, N)$.

(S,T)	$b(S,T)$	(S,T)	$b(S,T)$	(S,T)	$b(S,T)$
$(N,\emptyset)$	60	(A,B)	-15	$(AB,\emptyset)$	32
$(A,\emptyset)$	-15	(A,C)	-15	$(AC,\emptyset)$	-15
$(B,\emptyset)$	-15	(B,C)	-15	$(BC,\emptyset)$	-15
$(C,\emptyset)$	-15	(B,A)	-15	$(\emptyset,AB)$	-15
$(\emptyset,N)$	-15	(C,A)	-15	$(\emptyset,AC)$	-15
$(\emptyset,A)$	-15	(C,B)	-15	$(\emptyset,BC)$	-15
$(\emptyset,B)$	-15	(AB,C)	25	(C,AB)	-15
$(\emptyset,C)$	-15	(AC,B)	-15	(B,AC)	-15
$(\emptyset,\emptyset)$	0	(BC,A)	-15	(A,BC)	-15

Table IV: S.T.A.R. Company's Possible Profits with Star coalitions

We can compute the payoff for every (S,T) in the same manner. The payoffs are enumerated on Table **??**.

All the players of the star coalition must say 'yes' to get a better payoff for everyone. This highlights their 'elite' status in the game. In the context of the S.T.A.R. Company, the players belonging in the star coalition are the rich businessmen who can convince others. If at least one of them thinks that the project is not worthy of investment, then they set a role model for all the others to withdraw their money, which will result in net loss for the company.

If $S^\star \subseteq S$, then we call $S^\star$ a **strong star coalition**; otherwise, it is called a **weak star coalition**.

The above definition says that every player $p_m \in S^\star$ has a veto power so that when S does not contain all of $S^\star$ then (S,T) does not gain any positive reward. Moreover, the sum of all marginal contribution of all detractors does not exceed the 'excess' $b(N,\emptyset) - b(S,T)$ whenever $S^\star \subseteq N\backslash T$.

We consider a special case of a star coalition. If $|S^\star| = 1$ (i.e. $S^\star = \{p\}$) then p is called a **dictator**. In this case, p can behave as a single boss, where any coalition containing him gains a better payoff.

Using the theorems in the clan games above as inspiration, we formulate the following theorems below:

Theorem 3. *The core of a bicooperative game $b \in BG^N$ that has a star coalition $S^\star$ can be described by*

$$C(N,b) = \{x \in I(N,b) | x_i \leq M_i^d(N,b) \ \text{ for all } \ i \notin S^\star\}.$$

Theorem 4. *Let $b \in BG^N$ with $b(S,T) \geq b(\emptyset,N)$ for all $(S,T) \in 3^N$. If the following are satisfied:*

(i) $[b(N,\emptyset) - b(\emptyset,N)]e^j \in C(N,b)$ for all $j \in S^\star$ such that either $j \in S$ or $j \in T$ where $e^j \in \mathbb{R}^n$ of the form $e^j = (0,0,\ldots,\underbrace{1}_{jth},\ldots,0,0)$

(ii) $\exists x \in C(N,b)$ such that $x_i = M_i^d(N,b)$ for all $i \notin S^\star$

then b has a star coalition $S^\star$.

Theorem 5. *Let $b \in BG^N$ with $b(S,T) \geq b(\emptyset,N)$ for all $(S,T) \in 3^N$. If b has a star coalition $S^\star$ then*

$$[b(N,\emptyset) - b(\emptyset,N)]e^j \in C(N,b) \ \text{ for all } \ j \in S^\star \ \text{ such that either } \ j \in S \ \text{ or } \ j \in T.$$

5. SUMMARY AND CONCLUSION

A bicooperative game is different from a cooperative game in the sense that the classic cooperative game formulates coalitions by partitioning the players into two disjoint coalitions S and $N\backslash S$. On the other hand, bicooperative game focuses on the formulating two subsets of N which are S and T. Such formation leads to partitioning the set of all players into three sets: the 'yes', the 'no' and those who do not belong to either partition, which we call the 'abstain'. Another difference is the grand reward to be divided among all the players. In a cooperative game, the amount to be divided among all players is the net payoff $v(N)$, while in a bicooperative game, the amount to be divided is the net profit, which is the difference between the payoff when everyone says 'yes' and the payoff when everyone says 'no'.

The differences between a classic cooperative game and a bicooperative game are summarized below.

The concept a player's payoff being greater than or equal to his contribution is one of the base concepts of this study. This reflects the reality that the player would only opt to cooperate if he will gain better payoff.

The concept of the core of the bicooperative game give us an allocation schemes that a bicooperative game can have enabling us to define payoffs to all players.

With the notion of bicooperative games, the concept of clan games from classic cooperative game theory can be viewed in the perspective of a bicooperative game. This is what lead the researchers to formally define a star coalition. If there is a star coalition, then some solutions like the core can be defined. The necessary and sufficient conditions for the existence of a star coalition are also derived.

	Classical Cooperative Game	Bicooperative Game
Formation of coalitions	$S \subset N$	$(S,T) \vert S \cap T = \emptyset$; $S,T \subseteq N$
Characteristic Function	$v : 2^n \to \mathbb{R}$; $v(\emptyset) = 0$	$b : 3^n \to \mathbb{R}$; $b(\emptyset,\emptyset) = 0$
Grand Coalition	N	$(N,\emptyset)$
Grand Reward	$v(N)$	$b(N,\emptyset) - b(\emptyset,N)$
core	$\sum_{i \in S} x_i \geq v(S)$; $S \neq \emptyset$	$y(S)+z(N \setminus T) \geq b(S,T) - b(\emptyset,N)$
Concept of Clan Games	Clan Games	Star Coalitions

Table V: Differences between Cooperative and Bicooperative Games

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