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The Core Allocation in Sponsored Games

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Abstract:
One of the main problems in cooperative game theory is the fair division of rewards that are jointly obtained by the cooperating members of a team. In the case of a particular game, known as sponsored game which consists of two sets of players; the sponsors $S = \{s_i | 1 \leq i \leq k\}$ and the team players $T = \{t_j | 1 \leq j \leq p\}$, each sponsor $s_i \in S$ tries to bring cooperation on the set of team players. Cooperation is attained by giving a corresponding reward system $v_i \in S_v^i$ to the formed coalition $M \subseteq T$ such that $v_i : 2^T \rightarrow \mathbb{R}_{\geq 0}$ with $v_i(\emptyset) = 0$. Every team player $t_j \in T$ now decides to join or not to join in a coalition $M \subseteq T$. A formed coalition M will then receive a group reward of $V(M) = \sum_{i=1}^{k} v_i(M)$. Since the members of S and T act simultaneously, their decisions affect the benefits received by the team players, as well as the payoff of the sponsors. In this paper, we discuss some allocation schemes for the team players. Specifically, we consider schemes that are designed based on the concept of the core and the dominance core.

Keywords: sponsored games; allocations; team players; sponsors

1. INTRODUCTION

Situations involving parties with conflicting interest have been an issue over years. In 1928, a paper of John von Neumann entitled " On the Theory of Games of Strategy" was published and a branch of mathematics known as *game theory* came about. This theory deals with the analysis of game. It covers how decision makers must choose their strategies that will affect the interest of others. In addition, game theory has been explicitly applied and recognized in several fields. This is because, game theory tries to mathematically capture behavior in strategic situations.

One of the main issues in game theory under cooperative game is the fair division of group rewards among the members of the team. Each situation has its own content and therefore acceptable fair allocation methods may differ from situation to situation. If the concept of the fair allocation in each state is known, then it will be useful to resolve the conflict between cooperating parties. In addition, this will motivate the decision makers to choose their action to increase their payoff and affect the decision of other people.

Some of the well known allocation concepts have been studied to solve conflicts. The cost allocation method presented by Myerson in [5] used graph theory to study the cooperation structure in games. Together with fair allocation rules, the new concept on allocation which is closely related to Shapley value had been formulated. Similary, in [7], Okamoto studied the cost allocation rule under conflict situation by modeling these conflicts as minimum coloring game. In this paper, the core, the nucleolus and the Shapley value were considered as the allocation concepts. The paper of Tnimoto and Kita [3] discussed the allocation method in a case of local bus transportation in Japan. The study on nucleolus and its variants were the main focus of the method of fairness allocation. The fairness concept of allocating the goods among the members of the group that will be acceptable by all the cooperating members were presented in the above mentioned research papers.

In this study, fair allocation method will be designed under the concept of the sponsored game which was presented in the paper of E. Nocon entitled "On Strategies of Sponsored Games". In this paper, upon cooperation of the team players, the researcher is interested on what each team member can achieve. This paper aims to present a characterization of fair allocation of the rewards received by the team players in sponsored games that will be based on the concept of core and dominance core.

2. SPONSORED GAME

In the paper entitled "On Strategies of Sponsored Games" authored by Dr. Ederlina G. Nocon, a sponsored game was presented. In this game, there are two sets of players. These are referred to as the sponsors denoted by $S = \{s_i | 1 \leq i \leq k\}$ and the team players represented by $T = \{t_j | 1 \leq j \leq p\}$. This game is a cooperative game in the point of view of the team players and non cooperative game in the perspective of the sponsors.

From the set of team players T, its subset M is known as the coalition. The set T is the grand coalition and the empty set is the empty coalition. The collection of all subsets of T is given by 2^T. Each member of T has exactly two strategies or possible moves, to join or not to join in any coalition.

In this game, the intention of each sponsor is to bring cooperation among the members of T by giving a corresponding reward to the coalition formed from T. Every sponsor s_i can choose from his collection of reward systems S_v^i while each team player t_j chooses to join in a coalition $M \subseteq T$. We let v_i $(i = 1, ..., k)$ be the corresponding reward system to be given to the team players by sponsor s_i. The reward system is given by $v_i : 2^T \to \mathbb{R}_{\geq 0}$ with $v_i(\emptyset) = 0$ for which $v_i(M)$ is the amount received by coalition M from sponsor s_i. So, the set of allowable actions for every team player t_j is the power set of T. If the collection of rewards $V \in \prod_{i=1}^k S_v^i$, called **move**, is formed then the coalition M from the team players T receives a total payoff $V(M) = \sum_{i=1}^k v_i(M)$ offered by all the sponsors to coalition M. Then every team player must come up with an **action** $\alpha_j : \prod_{i=1}^k S_v^i \to 2^T$. This means that for a move V of all the sponsors, team player t_j chooses to join coalition $\alpha_j(V)$. Sponsor's move V bring on a desirable set of coalition for each team player that yields to a maximum payoff. Once a coaliton M is formed, $V(M)$ will be divided among each cooperating team players of M with an agreed allocation scheme a.

On the other hand, each sponsor aims to maximize their payoff while giving rewards to coalition $M \subseteq T$. The net payoff of a member of the sponsors S when he gives out the reward v_i is given by $b_i(M) = G_i(M) - v_i(M)$ where $G_i(M)$ is the gross payoff to the sponsor s_i once a coalition M has formed and $v_i(M)$ is the reward given by sponsor s_i to coalition M. These sponsors are making decision without consulting each other but it does not mean that they are competing to have the best payoff. The collection of moves of the sponsors will be formed since every sponsor has an idea of what he will achieve once a coalition is formed. This collection is given by

$$V^* = (v_1, v_2, ..., v_k) \in \prod_{i=1}^{k} S_v^i.$$

In the classical cooperative game theory, some of the known allocation concepts are the imputation set, reasonable set, core, dominance core, stable set, the Shapley value and the nucleolus. The allocation concept for the team players in the sponsored game will be identified in the next chapter of this paper. However, if the allocation concept was agreed upon by all the team players, we let $a^{V,M}$ be the allocation for all the members of the coalition M that yields from a move V of all the sponsors. Each team player t_j receives $a_{t_j}^{V,M}$ such that $V(M) = \sum_{t_j \in M} a_{t_j}^{V,M}$. We can use a_{t_j} for the allocation of each team player t_j if it is clear that it is from the pair (V, M). From here, we can describe the desirable set of coalitions as follows:

$$A(V, j) = \arg\max \left\{ a_{t_j}^{V,M} | M \subseteq T \text{ and } t_j \in M \right\}.$$

This means that each $t_j \in T$ has his own set A_j of choice functions. If every t_j works according to $\alpha \in \prod_{j=1}^p A_j$ having the goal of maximizing their allocations, then the set of team players T will be partitioned into a set of coalitions. Suppose $\mathscr{T}$ is a particular partitioning of T. Then,

$$v_i(\mathscr{T}) = \sum_{M \in \mathscr{T}} v_i(M)$$

is the amount given by sponsor s_i to $\mathscr{T}$.

Now, the allocation scheme used to divide the rewards fairly among the members of the team will be presented in the succeeding section.

3. THE CORE

One of the goals of cooperative game theory is to determine the allocation scheme that will yield to fair allocation of group rewards. This "fair" allocation depends on how it will be defined on a particular game. The concept of core and dominance core will be utilized to divide the group rewards for the team players. Note that our basic idea is that the set of team players T may cooperate by creating an agreement among themselves in forming a coalition for them to get big group rewards.

It is understandable that we require the payoff of t_j to be at least $V(t_j)$. For if $a_{t_j}^{V,\alpha_j} < V(t_j)$, it would be better for him to work alone than join the coalition. Moreover, it is also reasonable to consume all of $V(M_r)$ among all of its members in order to have an "efficient" allocation system. Thus, in each of the allocation schemes discussed in this paper, we assume satisfaction of the following conditions:

(i) The allocated amount $a_{t_j}^{V,\alpha_j}$ for the team player t_j for an action $\alpha_j(V)$ is at least as large as the amount he receives on his own, expressed as

$$a_{t_j}^{V,\alpha_j} \geq V(t_j) \qquad (\forall \quad t_j \in \alpha_j(V) \in \mathscr{T}_u). \tag{1}$$

(ii) The total allocation of the cooperating members of a coalition $M_r \in \mathscr{T}_u$ is equal to the sum of all the payoffs of the members of the coalition, written as

$$\sum_{t_j \in M_r} a_{t_j}^{V,M_r} = V(M_r). \tag{2}$$

For the reward V of all the sponsors, the set $I(V, M_r)$ contains an allocation that satisfies the above conditions for any $M_r \in \mathscr{T}_u$. This set is what we call as the **imputation** set for the coalition M_r. We denote by $I(V, \mathscr{T}_u)$ the set of imputations for the partitioning $\mathscr{T}_u$ of T given by

$$I(V, \mathscr{T}_u) = \Big\{ a = (a^{V,M_r})_{M_r \in \mathscr{T}_u} \mid a_{t_j}^{V,M_r} \geq V(t_j) \text{ and }$$

$$\sum_{t_j \in M_r} a_{t_j}^{V,M_r} = V(M_r) \text{ for all } t_j \in M_r \text{ and } M_r \in \mathscr{T}_u \Big\}. \tag{3}$$

In the core and dominance core allocation schemes, we assume that V define the reward of the sponsors including the formation of a partitioning $\mathscr{T}_u$ of T. We will then use the notation

$$a = (a^{V,M_r})_{M_u \in \mathscr{T}_u}$$

to denote an allocation vector so that $a_{t_j}^{V,M_r}$ refers to the payoff of team player t_j as a member of $M_r \in \mathscr{T}_u$.

The **core** for $(V, \mathscr{T}_u)$ is given by

$$C(V, \mathscr{T}_u) = \Big\{ a \mid a^{V,M_r} \in I(V, M_r) \text{ and }$$

$$\sum_{t_j \in W} a_{t_j}^{V,W} \geq V(W), \forall\, W \subseteq M_r, W \neq \emptyset \Big\}. \tag{4}$$

No subset of M_r will attempt to form a smaller coalition, so that M_r should stay intact. Observe that the core can also be described as follows

$$C(V, \mathscr{T}_u) = \Big\{ a \mid a^{V,M_r} \in I(V)$$

$$\text{and } e(W, a^{V,W}) \leq 0, \forall\, W \subseteq M_r \Big\} \tag{5}$$

where $e(W, a^{V,W}) = V(W) - \sum_{t_j \in W} a_{t_j}^{V,W}$.

In (5), the core is defined in terms of the value $e(W, a^{V,W})$ which we call as *excess*. This means that there is no positive excess for each $t_j \in M_r$ to have a better allocation.

From the set of imputations $I(V, M_r)$ with respect to reward V and a coalition M_r, let $a^{V,M_r}, b^{V,M_r} \in I(V, M_r)$ and $W \subseteq M_r$. We say that a^{V,M_r} dominates b^{V,M_r} via coalition W if

(i) $a_{t_j}^{V,M_r} > b_{t_j}^{V,M_r}$ for all $t_j \in W$ and

(ii) $\sum_{t_j \in W} a_{t_j}^{V,W} \leq V(W)$.

We use the notation $D(M_r, W)$ to denote all imputations that are dominated by some imputation a^{V,M_r} via $W \subseteq M_r$. The set

$$DC(V, M_r) = I(V, M_r) \setminus \cup_{W \subseteq M_r} D(M_r, W) \tag{6}$$

is called as the *dominance core* for a coalition M_r for a fixed reward V of all the sponsors. From this, we form another allocation scheme called the **dominance core** determined by the pair $(V, \mathscr{T}_u)$ given by

$$DC(V, \mathscr{T}_u) = \Big\{ a \mid a^{V,M_r} \in DC(V, M_r) \Big\} \tag{7}$$

In this allocation concept, no members of a subcoalition $W \subseteq M_r$ will get a dominated payoff from his set of possible allocation. This is because, $DC(V, \mathscr{T}_u)$ contains undominated imputations.

Theorem 1. *Let V be a move of all the sponsors and M_r be an action of each team player $t_j \in T$ where $M_r \in \mathscr{T}_u$. Then $C(V, \mathscr{T}_u) \subseteq DC(V, \mathscr{T}_u)$.*

Proof. It suffices to show that for all $M_r \in \mathscr{T}_u$, $C(V, M_r) \subseteq DC(V, M_r)$. Let $a^{V,M_r} \in C(V, M_r)$ such that $a^{V,M_r} \notin DC(V, M_r)$. Then there is a $b^{V,M_r} \in I(V, M_r)$ and a coalition $W \subseteq M_r$ such that $b^{V,W}$ dom_W $a^{V,W}$. Hence,

$$V(W) > \sum_{t_j \in W} b^{V,W}_{t_j} > \sum_{t_j \in W} a^{V,W}_{t_j}. \quad (8)$$

This implies that $V(W) > \sum_{t_j \in W} a^{V,W}_{t_j}$ which contradicts our assumption that $a^{V,M_r} \in C(V, M_r)$. This is because, for any $a^{V,M_r} \in C(V, M_r)$,

$$V(W) \leq \sum_{t_j \in W} a^{V,W}_{t_j}$$

for all $W \subseteq M_r$. Thus, $C(V, M_r) \subseteq DC(V, M_r)$. □

Theorem 1 implies that every member of $C(V, \mathscr{T}_u)$ is an undominated imputation.

4. CORE ALLOCATION WITH BARGAINING

We consider here an allocation scheme which allows "bargaining" among all the team players. In this system, we take a look at a problem on the structure of a coalition, that is, on the existence of a deviator in a group. Also, team players are allowed to give a part of their payoffs in order to convince other players to join them in their "favored coalition" resulting in a better payoff. Throughout this section, an allocation $a = (a_{t_j})_{M_r \in \mathscr{T}_u}$ of all the team players T is computed based on any of the allocation scheme including those that are presented in the previous section.

Basic Assumptions

Team players T will be partitioned in such a way that they will minimize their loss. Such partitioning of T is based on the following:

(i) Each team player $t_j \in T$ will compute his payoff based on the agreed allocation scheme of all the team players in T presented in the previous section .

(ii) A coalition M_r may be formed such that all of its members get their "best" payoffs.

(iii) Team players who will not achieve their maximum payoff will continue to search for the coalition that will give them their second best allocation.

(iv) Process (i)-(iii) will continue until all team players $t_j \in T$ have their coalition to join and a partitioning $\mathscr{T}_u \in P_T$ was formed.

In a partitioning $\mathscr{T}_u$ of T, a feasible payoff is defined as follows.

Definition 1. *Let $N \subseteq T$ and $b = (b^{V,N}_{t_m})_{t_m \in N}$. Then b is a* ***feasible payoff for*** *N* ***with respect to*** *$\mathscr{T}_u$ if for any $W \subseteq M_s$ and any $M_s \in \mathscr{T}_u$*

$$\sum_{t_m \in W} b_{t_m} \leq V(W).$$

In the long run, there might exist a team player $t_d \in T$ who may choose to deviate and form a coalition to achieve more through negotiation with other team players. We will call this team player a *deviator*.

Definition 2. *Let $t_d \in M_r$ with $M_r \in \mathscr{T}_u$, $a = (a^{V,M_r}_{t_j})_{M_r \in \mathscr{T}_u}$ be a payoff of team players T. Then t_d is called a* ***deviator*** *if there exists a coalition $N \subseteq T$ and a feasible payoff b for N with respect to $\mathscr{T}_u$ such that*

(i) $t_d \in N$;

(ii) $b^{V,N}_{t_d} > a^{V,M_r}_{t_d}$;

(iii) $b^{V,N}_{t_s} \geq a^{V,Q}_{t_s}$, *for all* $t_s \in N$ *such that* $t_s \in Q$ *and* $Q \in \mathscr{T}_u$.

We call coalition N as a ***prospect coalition*** *for deviator $t_d \in M_r$ and b the* ***objection of*** *t_d* ***to*** *M_r.*

Members of N can possibly divide their group reward $V(N)$ among themselves following the payoff plan according to b. Observe that t_d can convince $t_s (\in N)$ to leave his original group $Q \in \mathscr{T}_u$ because of (iii) and if this happens $\mathscr{T}_u$ "disintegrates" (that is, t_d leaves M_r and t_s leaves Q).

Definition 3. *Let $t_r \in M_r$ with $M_r \in \mathscr{T}_u$, $a = (a^{V,M_r}_{t_j})_{M_r \in \mathscr{T}_u}$ be a payoff of team players T. Then t_r is called a* ***defender*** *if there exists a coalition $N' \subseteq T$ and a feasible payoff c for N' with respect to $\mathscr{T}_u$ such that*

(i) $t_r \in N'$;

(ii) $c_{t_s}^{V,N'} > a_{t_s}^{V,Q}$ *for all* $t_s \in N'$ *such that* $t_s \in Q$ *and* $Q \in \mathscr{T}_u$ *;*

(iii) $c_{t_s}^{V,N\cap N'} \geq b_{t_s}^{V,N\cap N'}$*, for all* $t_s \in N \cap N'$*.*

We call a payoff vector c *as a* ***counter-objection of*** t_r ***to a deviator*** t_d*.*

In this concept, team player t_r is now referred to as a defender. This team player is a powerful member of M_r who can thwart $t_d{}'s$ plan of forming N. Hence, this shows that an existence of a deviator t_d is prevented in any coalition $M_r \in \mathscr{T}_u$. This results to a stability of the formation of a partitioning of T together with the allocation of each $t_j \in T$.

Definition 4. *A payoff* $a = (a_{t_j}^{V,M_r})_{M_r \in \mathscr{T}_u}$ *of each* $t_j \in M_r$ *is* ***stable*** *if for every objection* b *of deviator* t_d *there is counter objection* c *of defender* t_r*.*

This means that a payoff a is stable if for every deviator $t_d \in M_r$ there is a defender $t_r \in M_r$ who can prevent t_d in forming N.

Definition 5. *For a pair* $(V, \mathscr{T}_u)$*, a* ***pre-stable set*** *denoted by* ***preSS*** *is the set of all stable members of* $I(V, \mathscr{T}_u)$*.*

Theorem 2. *Let* $(V, \mathscr{T}_u)$ *be a pair of reward of all the sponsors and a partitioning* $\mathscr{T}_u$ *of* T*. Then*

$$C(V, \mathscr{T}_u) \subseteq preSS(V, \mathscr{T}_u).$$

Proof. Let $a \in C(V, \mathscr{T}_u)$ and suppose $t_d \in M_r$ is a deviator with prospect coalition N and objection b. Then $b_{t_d}^{V,N} > a_{t_d}^{V,M_r}$. Consider

$$W = \left\{ t_j \in M_r \cap N \mid b_{t_s}^{V,N} > a_{t_s}^{V,Q} \right\}.$$

We know that $W \neq \emptyset$ since $t_d \in W$ and $W \subseteq M_r$. Then for all $t_s \in W$, we have $b_{t_s}^{V,N} > a_{t_s}^{V,Q}$ and since b is a feasible payoff,

$$\sum_{t_s \in W} b_{t_s}^{V,N} \leq V(W).$$

This tells us that b dominates a via coalition $W \subseteq M_r$ which is a contradiction since $C(V, \mathscr{T}_u) \subseteq DC(V, \mathscr{T}_u)$. Therefore, every t_j member of M_r cannot be a deviator so that $a \in C(V, \mathscr{T}_u)$ is a stable member of $I(V, \mathscr{T}_u)$, that is, $a \in preSS(V, \mathscr{T}_u)$. Moreover, $C(V \mathscr{T}_u) \subseteq preSS(V, \mathscr{T}_u)$. □

This result shows that if a payoff a is contained in $C(V, \mathscr{T}_u)$ then a is stable.

5. CONCLUSION AND RECOMMENDATIONS

This paper characterizes core allocation scheme for a cooperative game known as the sponsored game $\langle S, T \rangle$. In this paper, the reward of each $s_i \in S$ is fixed so that a specific move V is considered by all team players T in calculating their payoffs. As expected, each aims to achieve his best payoff. From this, a partitioning of T is formed. Then, the core allocation was defined on this partitioning.

With the concept of bargaining, we see how a partitioning of T is affected by a team player t_j 's decision of forming such coalition. Moreover, the formation of a partitioning of T together with their core allocation becomes stable if for every objection of a deviator $t_d \in M_r$ there must exists a counter-objection of some $t_r \in M_r$, where $M_r \in \mathscr{T}_u$.

The following are recommended for further study:

1. design other allocation concepts of sponsored game using the concept of nucleolus and the Shapley Value.
2. determine whether the payoff a is stable if a is contained in the other allocation schemes.
3. present an extension of a result of this paper given that there is a set of deviators in T.

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