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Dynamical Modelling of Competing Firms

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Abstract: Competition is the heart and soul of enterprise. Majority of the everyday goods we consume are products of a small number of competing firms (oligopolies). Competition of firms each producing a differentiated product is usually modelled using either the Cornout model—the quantity approach—or the Bertrand model—the price approach, both of which provide alternative viewpoints on the supply side of the market. Both models are however static in the sense that they are used to find equilibrium or the steady state prices. In this study, a logistic adjustment process for price setting was applied as extension to the Bertrand model;this, as a result, yields dynamical equations similar to the Lotka-Volterra equations (LVE). This set of equations describe the "movement" of prices over time given specific values as proxies for brand strength, customer loyalty, level of costs, and other factors. Although LVE was initially used to model predator-prey systems, and then modified to describe ecological competition, this study uses the same concept to describe the case of economical competition. The resulting Lotka-Volterra dynamics and flow maps are then analyzed and plotted using Mathematica; after which they are analyzed for stability and for different situations, e.g. starting price, brand strength, customer loyalty. The results coincide with the static steady state case and exhibit a Nash equilibrium as postulated by mathematician and game theorist John Forbes Nash. Several conclusions arise from this competition model. First, the resulting Lotka-Volterra dynamics tend towards a steady state similar to the static case; second, price levels depend on brand strength, customer loyalty, costs, and not on company adaptability; and finally, the conditions by which companies will run out of business and exit the market are provided by the model.

Key Words: competition; duopoly; Bertrand; Lokta-Volterra; Nash

1. INTRODUCTION

Large companies compete in markets and the strategies they use dictate the price levels of the goods we buy. In this paper, a mathematical model inspired by the ideas of Bertrand, Lotka and Nash is formulated to model the movement ofprice levels, sales, and profits as a result of companies' adaptability, brand strength, and other factors. A different strategy—cooperative—is then explored and then compared with the default competitive scenario. Such aproach can be used in business analysis, research and development.

2. METHODOLOGY

1.1 The Competitive Case

Considering the Bertrand duopoly model for differentiated products, the direct demand functions of products of firms 1 and 2 are given by:

$$q_1 = A_1 - a_1p_1 + b_{12}p_2 \quad \text{(Eq. 1)}$$
$$q_2 = A_2 - a_2p_2 + b_{21}p_1 \quad \text{(Eq. 2)}$$

where:

q_i =consumption of product i
p_i = price of product i
A_i = overall brand strength of product i
a_i =own price effect on sales
b_{ij} = cross price effect on sales

Each firm's objective is to maximize profits. The profit of firm $i$is given by:

$$\pi_i = revenue_i - cost_i$$
$$= p_iq_i - (f_i + m_iq_i) \quad \text{(Eq. 3)}$$

where:

f_i = fixed costs of firm i
m_i = marginal (per unit) costs of firm i

Maximizing and assuming the other firm keeps its price constant (Bertrand conjecture: $\partial p_i/\partial p_j = 0$):

$$maximize\ \pi_i$$
$$subjectto p_1 \geq 0\ and p_2 \geq 0$$

$$\frac{\partial \pi_i}{\partial p_i} = 0; \quad i,j = 1,2; \quad i \neq j$$

$$= A_i + a_im_i + b_{ij}p_j - 2a_ip_i = 0$$

$$\Rightarrow p_i^R(p_j) = \frac{A_i + a_im_i}{2a_i} + \frac{b_{ij}}{2a_i}p_j \quad \text{(Eq. 4)}$$

If we let

$$\gamma_i = \frac{A_i + a_im_i}{2a_i} = \frac{\phi_i}{2a_i} \text{ and } \quad \beta_{ij} = \frac{b_{ij}}{2a_i}$$

then

$$p_i^R(p_j) = \gamma_i + \beta_{ij}p_j \text{for} \quad i = 1,2; \quad i \neq j \text{(Eq. 5)}$$

These are the price reaction functions (PRFs).For firm i to update its initial price p_{i0} to the desired level $p_i^R(p_j)$given in Eq. 5, it needs to have an adjustment process.

A logistic adjustment process is assumed with $p_i^R(p_j)$ as the asymptotic limit. The result is a dynamical model given by logistic equations:

$$\frac{dp_i}{dt} = r_i(p_i^R - p_i)p_i$$

where:

r_i = adaptability (scaling factor) of firm i

$$\Rightarrow \frac{dp_i}{dt} = r_ip_i(\gamma_i - p_i + \beta_{ij}p_j) \quad \text{(Eq. 6)}$$

Equation (11) is formally equivalent to the renowned Lotka-Volterra competition equations in the study of complex systems. The rate of growth r_i would pertain to the growth of product i's pricing; the carrying capacity would just be γ_i and would depend on the base price and the marginal cost faced by firm i; and the coefficient β_{ij} emerges as a result of differentiation among the products and self-price effect.

Solving for the equilibrium prices the steady state (equilibrium) prices, we get:

$$p_1^* = \frac{\gamma_1 + \beta_{12}\gamma_2}{1 - \beta_{12}\beta_{21}}; \quad p_2^* = \frac{\gamma_2 + \beta_{21}\gamma_1}{1 - \beta_{21}\beta_{12}}$$

1.2 The Cooperative (Collusive) Case

If the two firms collude and act as a monopoly, the profit maximization will be that of the shared profits:

$$maximize(\pi_1 + \pi_2)$$
$$subjectto p_1 \geq 0\ and p_2 \geq 0$$

Using a similar method to the previous case, the following differential equations are obtained:

$$\frac{dp_1}{dt} = r_1p_1(\gamma_1 - p_1 + \beta_{12}(2p_2 - m_2)) \quad \text{(Eq. 7)}$$

$$\frac{dp_2}{dt} = r_2p_2(\gamma_2 - p_2 + \beta_{21}(2p_1 - m_1)) \text{ (Eq. 8)}$$

Solving for the equilibrium prices the steady state (equilibrium) prices, we get:

$$p_1^* = \frac{(\gamma_1 - \beta_{12}m_2) + 2\beta_{12}(\gamma_2 - \beta_{21}m_1)}{1 - 4\beta_{12}\beta_{21}}$$

$$p_2^* = \frac{(\gamma_2 - \beta_{21}m_1) + 2\beta_{21}(\gamma_1 - \beta_{12}m_2)}{1 - 4\beta_{12}\beta_{21}}$$

3. RESULTS AND DISCUSSION

The evolution of the competition would vary depending on which condition the parameters of the system satisfy.

Expanding Eq. 5, profit π_i is:

$$\pi_i = (p_i - m_i)(A_i - a_ip_i + b_{ij}p_j) - f_i \text{(Eq. 9)}$$

Whereas market share μ_i:

$$\mu_i = \frac{revenue_i}{totalrevenues} = \rho_i / \left(\sum_{k=1}^{N} \rho_k\right) \qquad \text{(Eq. 10)}$$

where

$\rho_i = p_i q_i =$ revenue of firm i
$N =$number of competing firms

Note that for the steady state, prices and quantities sold approach equilibrium in the long run.

$$\text{Ast} \to \infty, (p_1, p_2) \to (p_1^*, p_2^*)$$

$$(q_1, q_2) \to (q_1^*, q_2^*) = \left(q_1(p_1^*, p_2^*), q_2(p_1^*, p_2^*)\right)$$

Note: For the following graphs, firm 1 will be plotted as blue and firm 2 as red for all graphs. The axes do not necessarily intersect at the origin.

Legend	
Firm 1 (competitive)	
Firm 2 (competitive)	
Firm 1 (collusive)	
Firm 2 (collusive)	

2.1 Competitive Advantage: Adaptability

The parameters r_1 and r_2 reflect the adaptability of firm 1 and firm 2 respectively. Fast adaptability may be a product of good management and/or research and development, the effects of which are represented by the parameters r_1 and r_2. A firm that has very good adaptability have a short term advantage in a competition, but the competitor eventually catches up in the long run. Figures 1a and 1b show the respective dynamics of prices and quantities sold having Firm 2 (red) as the more adaptive firm. The parameters are $r_1 = 1$ and $r_2 = 3$, setting all else equal (*ceteris paribus*).

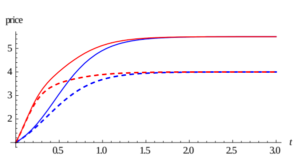

Figure 1a. Prices vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = m_2 = 1, r_1 = 1, r_2 = 3$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

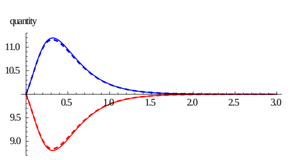

Figure 1b. Quantities vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = m_2 = 1, r_1 = 1, r_2 = 3$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

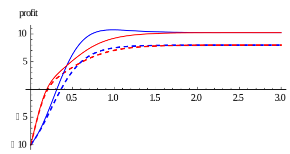

Figure 1c. Profits vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = m_2 = 1, r_1 = 1, r_2 = 3$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

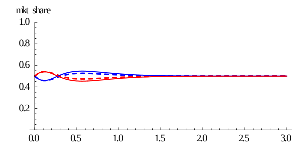

Figure 1d. Market Shares vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = m_2 = 1, r_1 = 1, r_2 = 3$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

Figures 1c and 1d show the respective dynamics of profits and market shares having. Firm 2 enjoys a higher profit and market share for a short while until firm 1 catches up. They approach the same equilibrium price in the long term which shows that adaptability r_i does not influence the long term outcome.

2.2 Competitive Advantage: Access to Low Costs and Production Efficiency

Costs or expenses come in two forms: fixed costs and marginal (per unit) costs. Production is said to be "efficient" if more output can be produced given the same input, this translates to a lower per unit cost of products. A firm will have a competitive advantage if either or both are lower than its competitor. This can be a result of having good suppliers, efficient operational management, and/or tax cuts. Figure 2a and Figure 2b show respective dynamics of prices and quantities with firm 2 (red) having access to lower fixed costs ($f_1 = 10$ and $f_2 = 5$). The lower fixed costs do not affect the set prices, and as a result, do not affect the quantities sold as well.

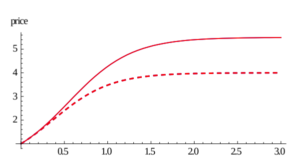

Figure 2a. Prices vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = m_2 = 1, r_1 = 1, r_2 = 1$, and

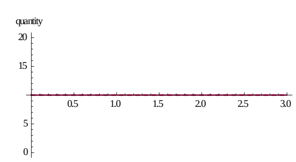

Figure 2b. Quantities vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = m_2 = 1, r_1 = 1, r_2 = 1,$

$f_1 = 10, f_2 = 5$. Initial prices: $p_1(0) = p_2(0) = 1$

and $f_1 = 10, f_2 = 5$. Initial prices: $p_1(0) = p_2(0) = 1$

Figures 2c and 2d show respective dynamics of profits and market shares. The lower fixed costs allow firm 2 to enjoy a higher profit despite selling at the same price, but the market shares still remain the same.

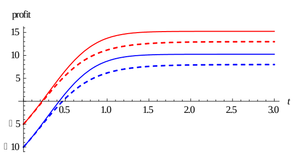

Figure 2c. Profits vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = m_2 = 1, r_1 = 1, r_2 = 1$, and $f_1 = 10, f_2 = 5$. Initial prices: $p_1(0) = p_2(0) = 1$

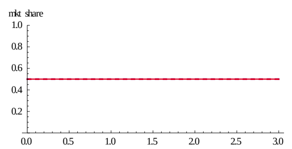

Figure 2d. Market Shares vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = m_2 = 1, r_1 = 1, r_2 = 1$, and $f_1 = 10, f_2 = 5$. Initial prices: $p_1(0) = p_2(0) = 1$

Figures 3a and 3b show the respective dynamics of prices and quantities sold with firm 2 having access to lower marginal costs ($m_1 = 2$ and $m_2 = 1$, all else equal). The lower marginal costs allow firm 2 to set a lower per unit price, thus attracting more buyers.

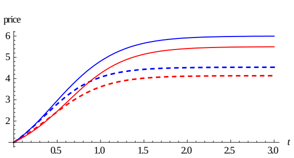

Figure 3a. Prices vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = 2, m_2 = 1, r_1 = r_2 = 1$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

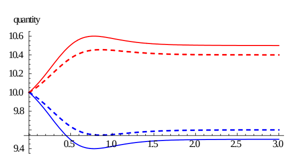

Figure 3b. Quantities vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = 2, m_2 = 1, r_1 = r_2 = 1$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

Figures 3c and 3d show the respective dynamics of profits and market shares. The lower marginal costs allow firm 2 to enjoy a higher profit, as well as higher market share in the long run despite selling at a lower price.

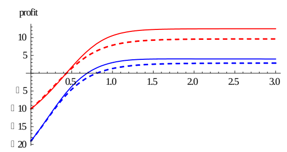

Figure 3c. Profits vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = 2, m_2 = 1, r_1 = r_2 = 1$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

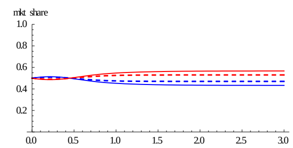

Figure 3d. Market Shares vs. Time
$A_1 = A_2 = 10, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = 2, m_2 = 1, r_1 = r_2 = 1$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

2.3 Competitive Advantage: Brand Strength, Loyalty, and Demand

A brand is said to be "strong" or "in demand" if many people buy it. Customers are said to be "loyal" if they patronize a certain brand and are unwilling to change brands. This is often related to product quality/design and good marketing/advertising strategies. In the direct demand functions, the effect brand strength or demand is portrayed by the parameter A_i and the cross price parameter b_{ji}.

The parameter A_i reflects product i's overall brand strength—the higher A_i is, the higher is the demand for q_i for all levels of price. Figures 4a and 4b show the movement of prices and quantities sold, respectively, over time with Firm 2 (red) having a greater overall brand strength $A_1 = 10$ and $A_2 = 15$. Its prices and quantities sold are consistently higher.

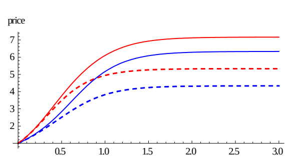

Figure 4a. Prices vs. Time
$A_1 = 10, A_2 = 15, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

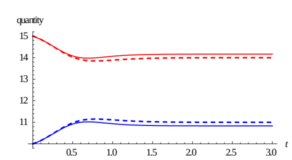

Figure 4b. Quantities vs. Time
$A_1 = 10, A_2 = 15, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1$, and $f_1 = f_2 = 10$. Initial prices: $p_1(0) = p_2(0) = 1$

Figures 4c and 5d show the movement of profits and market shares, respectively, over time. Like that of prices and quantities sold, the profits and market shares of firm 2 are consistently higher due to having an overall strong brand.

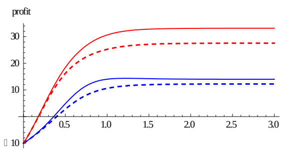

Figure 4c. Profits vs. Time

$A_1 = 10, A_2 = 15, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

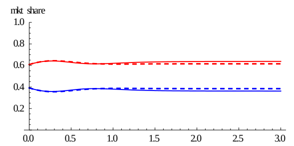

Figure 4d. Market Shares vs. Time

$A_1 = 10, A_2 = 15, a_1 = a_2 = 2, b_{12} = b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

On the other hand, b_{ji} measures the transferring of consumers from q_j to q_i when p_j increases. The value of b_{ij} is dependent on the customer loyalty and the degree of similarity/substitutability of products. When a certain firm i has a high degree of customer loyalty, the value of b_{ij} will be very small—consumers are less likely to transfer. When both firms sell products that are very similar/substitutable, the values of b_{ij} and b_{ji} will be high—consumers are not that particular about the brand due to similarity.

Figures 5a and 5b illustrate the case when b_{12} and b_{21} are asymmetric. Suppose firm 2 has stronger customer loyalty relative to firm 1 such that $b_{12} = 0.5$ and $b_{21} = 1$. Firm 2 enjoys the privilege of setting a higher price without suffering in quantities sold.

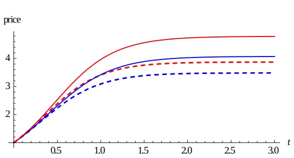

Figure 5a. Prices vs. Time

$A_1 = 10, A_2 = 10, a_1 = 2, a_2 = 2, b_{12} = 0.5, b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

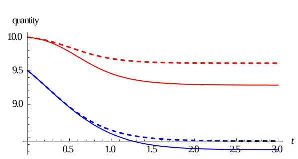

Figure 5b. Quantities vs. Time

$A_1 = 10, A_2 = 10, a_1 = 2, a_2 = 2, b_{12} = 0.5, b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

Figures 5c and 5d show that firm 2, having stronger customer loyalty, naturally enjoys the advantage of having higher profit and market share.

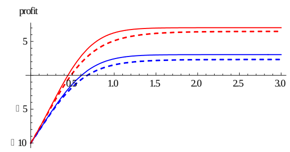

Figure 5c. Profits vs. Time

$A_1 = 10, A_2 = 10, a_1 = 2, a_2 = 2, b_{12} = 0.5, b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

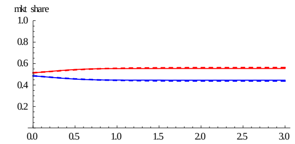

Figure 5d. Market Shares vs. Time

$A_1 = 10, A_2 = 10, a_1 = 2, a_2 = 2, b_{12} = 0.5, b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

2.4 Competitive Advantage: Effusion

Price is a major factor in a consumer's decision making regarding which products to buy. In general, people are less willing to buy expensive goods especially when cheaper alternatives are available—they "effuse" from patronizing a certain brand as its price increases. High customer loyalty lessens effusion. Such effect are mirrored in the parameter a_i (own price effect); the worse the effusion of firm i's consumer base, the higher the value of a_i.

Figures 6a and Figure 6b illustrate the respective differences in prices and quantities sold when firm 2 experiences worse effusion relative to firm 1. The parameters are $a_1 = 2$ and $a_2 = 2.5$, setting all else equal. Firm 2 is forced to lower its prices to sell more.

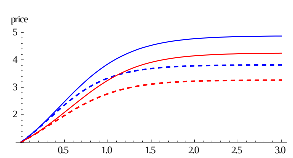

Figure 6a. Prices vs. Time

$A_1 = 10, A_2 = 10, a_1 = 2, a_2 = 2.5, b_{12} = 0.5, b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

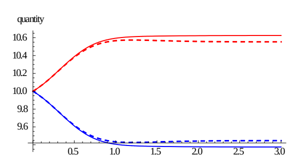

Figure 6b. Quantities vs. Time

$A_1 = 10, A_2 = 10, a_1 = 2, a_2 = 2.5, b_{12} = 1, b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

Figures 6c and 6d illustrate the respective differences in profits and market shares between the firms. Firm 2, despite selling more, still experiences lower profit and market share.

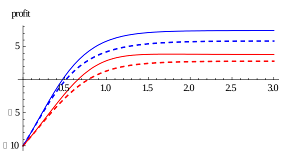

Figure 6c. Profits vs. Time
$A_1 = 10, A_2 = 10, a_1 = 2, a_2 = 2.5, b_{12} = 0.5, b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

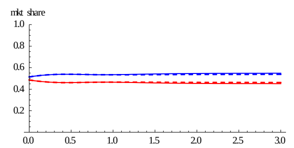

Figure 6d. Market Shares vs. Time
$A_1 = 10, A_2 = 10, a_1 = 2, a_2 = 2.5, b_{12} = 1, b_{21} = 1, m_1 = 1, m_2 = 1, r_1 = r_2 = 1,$ and $f_1 = f_2 = 10.$ Initial prices: $p_1(0) = p_2(0) = 1$

2.5 Phase Portraits

Figures 7a is typical phase portrait of a symmetric—in terms of parameters—competitive case. The red point is the steady state value for prices. Figure 7b shows firm 1 having greater adaptability parameter; this changes the "flow" of the prices towards equilibrium.

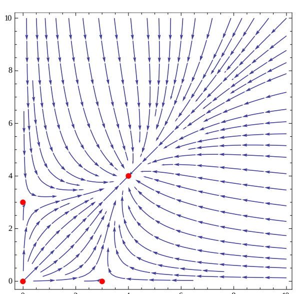

Figure 7a. Flow Map
$\gamma_1 = 3, \gamma_2 = 3, \beta_{12} = 0.25, \beta_{21} = 0.25,$
$r_1 = 1, r_2 = 1$

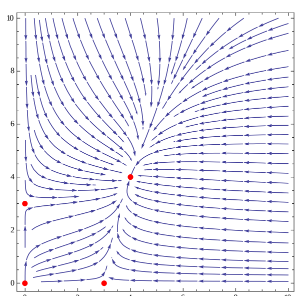

Figure 7b. Flow Map
$\gamma_1 = 3, \gamma_2 = 3, \beta_{12} = 0.25, \beta_{21} = 0.25,$
$r_1 = 3, r_2 = 1$

For both figure 8a and 8b, a different value of gamma is plotted. They are symmetric due to the form of the dynamical equations.

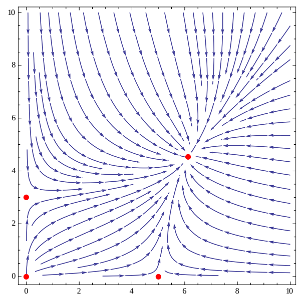

Figure 7a. Flow Map
$\gamma_1 = 5, \gamma_2 = 3, \beta_{12} = 0.25, \beta_{21} = 0.25,$
$r_1 = 1, r_2 = 1$

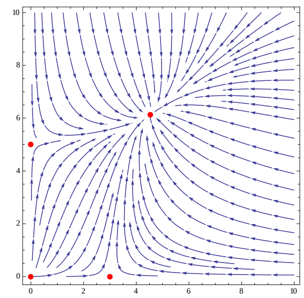

Figure 7b. Flow Map
$\gamma_1 = 3, \gamma_2 = 5, \beta_{12} = 0.25, \beta_{21} = 0.25,$
$r_1 = 1, r_2 = 1$

4. CONCLUSIONS

It is apparent that the strategy that gives the highest profit is to collude rather than to compete. The steady state values are shown in figure 7.

	Competitive				Collusive			
Fig.	p_1^*	p_2^*	π_1^*	π_2^*	p_1^*	p_2^*	π_1^*	π_2^*
1	4	4	8	8	5.5	5.5	10	10
2	4	4	8	13	5.5	5.5	10	15
3	4.5	4.1	2.8	9.6	6	5.5	4	13
4	4.3	5.3	12	28	6.3	7.2	14	33
5	3.5	3.9	2.3	6.5	4.1	4.8	3.1	7
6	3.8	3.3	5.9	2.8	4.9	4.3	7.4	3.8

Figure 7. Competition vs. Collusion

This study also shows that adaptability only result to short term advantages.Finally, brand strength and lower costs post and advantage over the long term—this suggests that companies invest in marketing/advertising and research and development.

5. ACKNOWLEDGMENTS

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