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The Game Chromatic Number of Some Classes of Graphs

Casper Joseph Destacamento[1], Andre Dominic Rodriguez[1] and Leonor Aquino-Ruivivar[1,*]
[1] *Mathematics Department, De La Salle University*
**leonor.ruivivar@dlsu.edu.ph*

Abstract: The *coloring game* is played by two players called Alice and Bob on the vertices of a graph G as follows: Using colors from a set $X = \{1, 2, \ldots, k\}$ of k distinct colors, the players take turns in assigning colors to the vertices of G such that no two adjacent vertices will receive the same color. The two players play alternately with Alice always moving first. The game ends when either all the vertices have been colored, or it is no longer possible to color an uncolored vertex. Alice wins in the first case, and Bob wins otherwise.

The *game chromatic number* of a graph G is a graph invariant representing the smallest number of colors for which Alice has a guaranteed winning strategy. The game chromatic number of a graph G is denoted by .

The game chromatic number of various classes of graphs, including trees, cactuses and cartesian products of various types of graphs, have been determined. In this paper, the game chromatic number of some common classes of graphs, such as paths, cycles, complete graphs, complete bipartite graphs,star graphs, fans, wheels, Cartesian product graphs, and the Petersen graph, are determined. Some of these results are established by using the relationship between the game chromatic number with another graph invariant called the *game coloring number*. A previously published result is also modified.

Key words: coloring game; game chromatic number; cartesian product graph; graph invariant; marking game; game coloring number

1. INTRODUCTION

By a ***graph*** we mean a pair G = (V(G),E(G)) where V(G) is a nonempty set of elements called the *vertices* of G, and E(G) consists of unordered pairs called *edges* of elements of V(G). The numbers |V(G)| and |E(G)| are called the *order* and the *size* of the graph G, respectively.

The coloring game is a game played by two players, who will be referred to as Alice and Bob, on the vertices of a finite graph G. Using colors from a set X = {~1,2, …, k}, a player makes a move by coloring an uncolored vertex of G such that no two vertices will be assigned the same color. Alice and Bob move alternately throughout the game, with Alice always making the first move. The game ends when no player can make a feasible move. Alice wins if all the vertices have been colored, and Bob wins otherwise.

The *game chromatic number* of the graph G, denoted by $\chi_g(G)$, is the smallest number of colors for which Alice has a winning strategy. The game assumes that the players are both rational and will always make the best possible move at any time in the game.

Example 1.
Consider the path P_3 order 3. Clearly, a set with only one color will not allow Alice to win. On the other hand, if there are two colors, then Alice can make her first move on the middle vertex, and the remaining two vertices can be assigned the second available color. This shows that $\chi_g(P_3) = 2$.

The *chromatic number* of a graph G is the smallest positive integer $\chi(G) = k$ such that the vertices of G can be assigned k colors such that adjacent vertices always receive different colors. It is clear that for any finite graph G, we have $\chi_g(G) \geq \chi(G)$. On the other hand, a related graph invariant called the *game coloring number* can be shown to be an upper bound for $\chi_g(G)$. Exact values or bounds for the game chromatic number of various classes of graphs have been determined. In (Bartnicki, et. al, 2008) and (Sia, 2009) exact values for the game chromatic numbers of cartesian product graphs involving paths, cycles, stars, wheels, complete graphs and complete bipartite graphs were found. In (Faigle et.al., 1993) the game chromatic numbers of trees and interval graphs was investigated, while in (Sidorowicz , 2007) the game chromatic number of families of cactuses was established.

In this paper, we determine the exact values of the game chromatic number of the following classes of graphs: paths, cycles, stars, wheels, complete graphs, complete bipartite graphs, and the Petersen graph. We also modify a result on the game chromatic number of the cartesian product graph of paths and wheels from (Sia, 2009).

2. RESULTS

2.1 The Game Chromatic Number of Some Simple Families of Graphs

In this section, we identify the game chromatic numbers of some familiar classes of graphs. These are: the paths, cycles, stars. Wheels, the complete graphs and the complete bipartite graphs.

The *path of order n*, denoted by P_n, is a walk of length n-1with n distinct vertices. The *cycle graph of order n*, denoted by C_n is the graph with vertex set $V(C_n) = \{ v_1, v_2, \ldots v_n \}$ such that $v_1, v_2, \ldots, v_n$ is a path, and $v_1 v_n$ is an edge. A *star of order n+1*, denoted S_n, is the graph with vertex set $V(S_n) = \{ \{ v_1, v_2, \ldots v_n \}$ and edge set $E(S_n) = \{ v_0 v_i$, i = 1, 2, …, n $\}$ The vertex v_o, usually called the *central vertex*, is adjacent to all the other vertices of the vertex set while the degree of each of the remaining n vertices is 1. A *wheel of order n+1*, denoted by W_n, is the graph with vertex set $V(W_n) = \{ v_1, v_2, \ldots v_n \}$, such that $v_1, v_2, \ldots, v_n$ form a cycle and v_o is adjacent with v_i, i = 1, 2, …, n. A *complete graph of order n*, denoted by $K_{n,}$ is a graph with n vertices which are pairwise adjacent. A *complete bipartite graph of order m+n*, denoted by $K_{m,n}$, is the graph satisfying the three conditions:

(a) The graph has vertex set $V(G) = A \cup B$, where A and B are disjoint non-empty sets,
(b) Both A and B are independent sets, and
(c) Every vertex in A is adjacent to each vertex in B.

Our results for these classes of graphs are given in the following propositions:

Proposition 1.
Let $P_n$be a path, of order $n \geq 2$ Then

$$\chi_g(P_n) = \begin{cases} 2 & if\ n = 2\ or\ 3 \\ 3 & otherwise \end{cases}$$

Proof:

For n = 2, it is clear that Alice always wins. If n = 3, Alice's winning strategy is to color the middle vertex first. For $n \geq 4$, if there are only two colors, then Bob can force a win by coloring a vertex at a distance of 2 from the vertex colored by Alice, thereby forcing a third color on their common neighbor. If there are 3 colors, then since each vertex is adjacent to at most 2 vertices, Alice is guaranteed to win no matter where she makes her first move.

Proposition 2.
Let C_n be a cycle of order $n \geq 3$. Then $\chi_g(C_n) = 3$.

Proof:
Similar to the path graphs, regardless of where Alice moves, she is guaranteed to win with 3 colors since every vertex is adjacent to 2 vertices.

Proposition 3.
For any star $S_n, n \geq 2$, we have $\chi_g(S_n) = 2$.

Proof:
Showing that Bob can win with one color no matter what Alice does is trivial. We now show that Alice can win with two colors. In any given star S_n, Alice's

strategy is to color the central vertex on her first move. The subsequent moves by Alice or Bob is to color the outer vertices with the second color. Since the outer vertices form an independent set, this second color is sufficient to color all of them. Therefore, $\chi_g(S_n) = 2$.

Proposition 4.
For any wheel $W_n, n \geq 3$, we have $\chi_g(W_n) = 4$.

Proof:
As with Alice's strategy in the star graph, she colors the central vertex in her first turn. The remaining uncolored vertices form a subgraph of W_n, isomorphic to that of a cycle of order \$n\$, which has a game chromatic number of 3. Then $\chi_g(W_n) = 4$.

Proposition 5.
Let W_n be a complete graph of order \$n\$. Then $\chi_g(K_n) = $ n.

Proof:
Since the vertices of K_n are pairwise adjacent, no two vertices can receive the same color. Thus $\chi_g(K_n) = $ n.

Proposition 6.
Let $K_{m,n}$ be a complete bipartite graph of order m+n. Then

$$\chi_g(K_{m,n}) = \begin{cases} 2 & if \min(m,n) = 1 \\ 3 & otherwise \end{cases}$$

Proof:
Since the graph is bipartite, the vertex set partitions into two independent sets A and B, with |A| = m,~|B| = n. We consider the following cases:

Case 1: min(m,n) = 1
Without loss of generality, assume |A| = 1. Alice forces a win by coloring the only vertex in A. Since the vertices in B are independent, one color is sufficient to color all of them.

Case 2: $m \geq 2, n \geq 2$
Since every vertex in A is adjacent with every vertex in B, it is clear that at least two colors must be used. Moreover, since $m \geq 2, n \geq 2$, Bob can force one of the two sets A and B to be colored with two colors. This shows that $\chi_g(K_{m,n}) \geq 3$. Without loss of generality, assume that Alice chooses to color a vertex in A with 1. If Bob responds to this move by coloring a vertex in the same independent set with a different color, then Alice responds by using the third color on the other independent set, which forces the graph to only 3 colors. On the other hand, if Bob colors on the other independent set, then we can see that regardless of where Alice moves, she will win the game given three colors.

In both cases, Alice can force a win with three colors. Hence, $\chi_g(K_{m,n}) = 3$.

2.2 The Game Chromatic Number of the Petersen Graph

The Petersen graph is the graph illustrated in Figure 1:

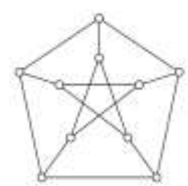

Figure 1. The Petersen graph

If we denote the Petersen graph by π, then the game chromatic number of this graph is given in the following result:

Theorem 1:
If π is the Petersen graph, then $\chi_g(\pi) = 3$.

Proof:
The proof consists of two parts, namely:
(a) Show that Bob can force a win with less than three colors.
(b) Show that Alice has a winning strategy when three colors are used.

Without loss of generality, suppose Alice colors in the outer cycle of the graph. Bob then colors a vertex of distance 2 from the vertex that Alice colored. The two colored vertices are adjacent to another vertex, indicated by an asterisk (*), which now requires a third color. Thus, Bob wins when there are only two playable colors. This is illustrated in Figure 2.

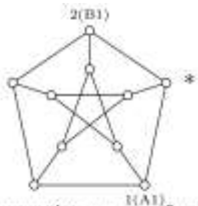

Figure 2. The successive moves force a third color

The successive moves show that a third color is necessary. Thus, $\chi_g(\pi) \geq 3$.

To show that Alice has a winning strategy with three colors, we fix Alice's initial move to be on one of the vertices of the outer cycle. We show that with this initial move, no matter how Bob plays, Alice will always be able to force a win.

We consider five cases corresponding to Bob's first move. These are illustrated in Figure 3 below. Vertices that have been colored are labeled as ``color(player turn)''. For example, 1(A1) means Alice used the color 1 on her first turn. The shaded vertices indicate where Bob will make his first move. When there are two shaded vertices in the graph, it means that the two vertices are symmetric with respect to the vertex colored by Alice, and Bob can choose either vertex to color.

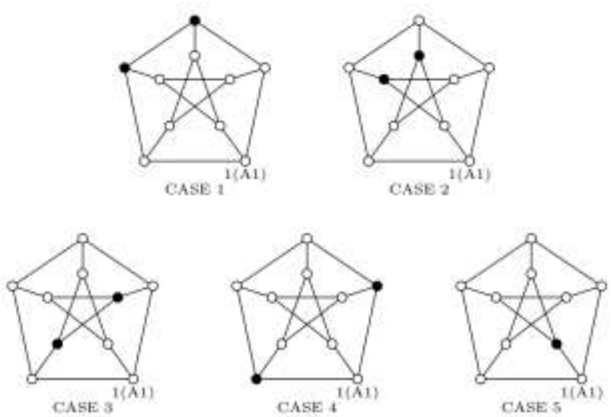

Figure 3. The 5 cases for the Petersen graph

Each of these cases can be subdivided into several subcases showing Alice's second move. The proof is completed by exhausting all possible moves and countermoves by Alice and Bob. In all of these, it can be shown that Alice is able to prevent a vertex from having three different colored neighbors, and hence limit the play to three colors. This shows that $\chi_g(\pi) = 3$.

2.3 The Game Chromatic Number of Cartesian Product Graphs

Recall that if G_1 and G_2 are graphs, then the cartesian product $G_1 \times G_2$ is the graph with vertex set $V(G_1) \times V(G_2) = \{(u,v) | u \in V(G_1), v \in V(G_2)\}$ and edge set $E(G_1 \times G_2) = \{[(u_1,v_1),(u_2,v_2)], u_1,u_2 \in V(G_1), v_1,v_2 \in V(G_2)$ such that either $u_1 = u_2$ and $(v_1,v_2) \in E(G_2)$, or $(u_1,u_2) \in E(G_1)$ and $v_1 = v_2\}$

Example 2.
Consider the path, $P_2$and the cycle, C_3, with vertex sets $\{x_1, x_2\}$ and$\{y_1, y_2, y_3\}$ respectively. The Cartesian product $P_2 \times C_3$ is pictorially represented and shown in Figure 4:

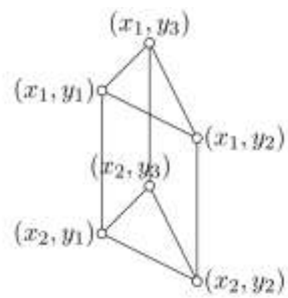

Figure 4: The graph $P_2 \times C_3$

In (Sia, 2009), the following result on the Cartesian product of the path $P_2$with the wheels W_n was given:

Theorem 2:
For any integer $n \geq 9$, $\chi_g(P_2 \times W_n) = 5$.

We will show that the above theorem holds for $n \geq 5$. In order to do this, we will use the concept of the game coloring number. Let G be a graph.

- A *linear order* L on G is any ordering of the vertices of G.
- Let L be a linear order on G and let x be a vertex of G. The *back degree of x relative to the linear order L*, denoted by $b_L(x)$, is defined by
$$b_L(x) = |\{y \in V(G) : xy \in E(G), y \text{ precedes } x \text{ in } L\}|$$

If L$ is a linear order on the vertex set of a graph G, then the *back degree function of L*, denoted by $b(L)$, is defined by $b(L) = \max\{ b_l(x) : x \in V(G)\}$.

We now define the *game coloring number* of a graph G. We consider a simpler two-person game on a graph G called the *marking game*. As before, we name the two players as Alice and Bob. The two players take turns marking vertices in G, with Alice making the first move. Only previously unmarked vertices can be chosen by a player for marking. A linear order L is induced by this game by arranging the vertices of G in the order in which they were marked. Alice's objective is to minimize $b(L)$, while Bob seeks to maximize it. At the start of the game, the two players decide on a positive integer k. Bob wins if at some point in the game, there exists an unmarked vertex v with k marked neighbors. On the other hand, Alice wins if this situation never arises. The *game coloring number* of G is the smallest integer k for which Alice has a winning strategy. This will happen if Alice can keep the number of marked neighbors of any vertex of G to less than or equal to k-1.

We formalize the definition of the game coloring number as follows. Let G be a graph, and let $k = \min\{ b(L): L\ is\ a\ linear\ order\ in\ G \}$ The *game coloring number* of G, denoted by $col_g(G)$, is defined to be $col_g(G) = k + 1$ where $k = \min\{ b(L) | L\ is\ a\ linear\ order\ in\ G \}$.

Note that for a given linear order L on the vertex set of a graph G, the back degree of any vertex x in G represents the number of vertices adjacent to x which were marked prior to x. If we relate this to the coloring game by interpreting the marked vertices to be the vertices that have already been colored, then the back degree of x can be interpreted as the number of neighbors of x that have been colored, using at most $b_L(x)$ colors. Hence, to color x and its neighbors, at most $b_L(x) + 1$ colors are needed. Therefore, the number *b(L) + 1* is an upper bound for the number of colors needed so that all the vertices of G can be colored, and $col_g(G) = k + 1$ is an upper bound for the least number of colors needed for Alice to win the coloring game. We have thus shown that $\chi_g(G) \leq col_g(G)$.

The following result which appeared in (Sia, 2009) will be useful in establishing the modification of Theorem 2.

Theorem 3: Let G and H be two graphs. Then
$\chi_g(G \times H) \leq col_g(G \times H) \leq col_g(\cup V(H))G + \Delta(H)$
where $(\cup V(H))G$ is the union of all *G*-fibers and $\Delta(H)$ is the maximum degree of a vertex in H.

The proof of Theorem 2 as it appears in (Sia, 2009) consists of describing a winning strategy for Alice, and the author remarked that the condition was necessary at one point in the strategy. However, while going over the proof, we failed to see the necessity of the above condition. Our investigations showed that the range of values for n can be expanded, and we have the following result.

Theorem 4:
For any integer $n \geq 5$, we have $\chi_g(P_2 \times W_n) = 5$. .

Proof:
There are three cases to consider for the proof, namely $n \geq 7, n = 6\ and\ n = 5$. For the latter two cases, proofs by exhaustion were used by considering the different possible cases for Bob's first move (distinguished by what vertex he could color and what color he will use in that vertex), and how Alice will respond to these moves.

Case 1: $n \geq 7$
For $n \geq 7$, a specific strategy was used for Bob to force the use of five colors for the graph. We first show that at least 5 colors are necessary for Alice to have a winning strategy. Without loss of generality, we will assume that Alice's first move is on a_0, and Bob's first move is on of the outer vertices of the second copy of W_n.

- If Alice colors b_0 with a third color, then Bob responds by coloring a_2 with a fourth color, creating two threats which Alice cannot block simultaneously, thus forcing the graph to five colors.

We will partition the remaining cases into two, the case where Alice makes her second move in the vertices a_i and the case when she makes her second move on any of the vertices b_i.

- Suppose Alice colors on the a_i's on her second move.. Without loss of generality, suppose she colors on either a_1 or a_n. Then Bob responds by coloring b_4 with color 3. Alice must now use a fourth color on b_0, for otherwise Bob can force a fifth color on his next turn. Assume Alice colors b_0 with color 4 and Bob responds to this move by coloring a_5 with color 2. This will create simultaneous threats to b_5 and to a_4. Alice cannot block both these threats so the graph is forced to five colors. If instead Alice colors on any of the vertices $a_i,\ i = \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n-1$, then Bob responds by coloring b_2 with color 3. Again Alice uses color 4 on b_0,, and Bob then responds by using color 2 on a_3, thus creating two threats to a_2 and to b_3. This again forces a fifth color on the graph.

- Suppose now that Alice colors one of the b_i's on her second move. We note that Alice will not use a third color on the b_i's since this would allow Bob to force a fifth color on his next turn. Hence, Alice's choices are either to use color 1 or color 2 on her second move. If Alice uses color 1 on any vertex b_i, then Bob responds by using color 3 on a vertex with distance 2 from the vertex that Alice colored. Thus, if Alice colored b_2, then Bob colors b_4. Again Alice must block the threat to b_0, so she uses a fourth color on this vertex. Her move creates a threat to vertex b_3, and Bob responds to this by coloring a_3 with the fourth color, forcing a fifth color to the graph. Suppose instead that Alice uses color 2 on a feasible vertex. Bob responds by using color 3 on a vertex with distance 2 from the vertex colored by Alice on her second second move. Thus, for example, if Alice colored vertex b_3, then Bob responds by coloring b_5. Again, Alice responds by coloring b_0 with a feasible color, and Bob responds to Alice's third move by color a_6 with color 2. This creates simultaneous threats to a_5 and to b_6, which Alice cannot both block, so the graph is forced to 5 colors.

We have thus shown that $\chi_g(P_2 \times W_n) \geq 5$ for $n \geq 7$. A pictorial representation of $P_2 \times W_7$ (with some edges removed for less confusion, but it is understood that these edges exist) is shown in Figure 5 below. Any of the cases above can be shown to work on this graph.

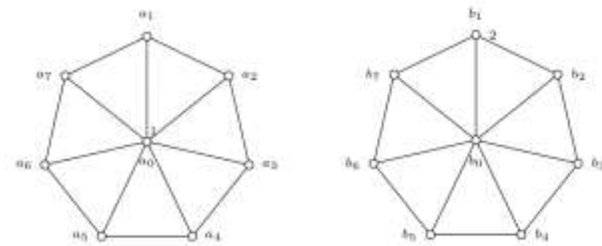

Figure 5. The graph of $P_2 \times W_7$

To show that $\chi_g(P_2 \times W_n) \leq 5$, we apply Theorem 3 with $G = W_n$ and $H = P_2$. Combining these two inequalities gives us $\chi_g(P_2 \times W_n) = 5, n \geq 7$.

Case 2: $n = 6$
For $n = 6$, the following six subcases to be considered are the possible moves that Alice can make on her second turn (In the figures below, these are denoted by the shaded vertices). For each of these subcases, the subsequent optimal moves by the two players will be considered. In each case, it can be shown that $\chi_g(P_2 \times W_6) = 5$.

Case 2: $n = 5$
As with $n = 6$, we identify all possible subcases based on Alice's first move (these are again denoted by the shaded vertices in the figures below) . The analysis is similar to those for the subcases in Case 2, so we will just identify the cases and their final configurations, as shown below:

This completes the proof of Theorem 4.

3. CONCLUSIONS

In this study, the game chromatic numbers of some classes of graphs were identified. Aside from the more common classes of graphs , such as paths, cycles, stars, wheels, complete graphs and complete bipartite graphs, results for the game chromatic number of the Petersen graph and the Cartesian product of paths with wheels were also presented.

The study of the game chromatic number of a graph is far from complete. For the Cartesian product graph alone, there is still no general result for the Cartesian product of two arbitrary graphs G and H. The determination of the game chromatic number of graphs obtained through various graph operations offers a fertile source of problems for future studies.

4. REFERENCES

Andres, S. D. and Hochstttler, W., (2011), The Game Chromatic Number and the Game Colouring Number of Classes of Oriented Cactuses, Information Processing Letters, 111, 222-226.

Bartnicki, T., et. al. (2008), Game Chromatic Number of Cartesian Product Graphs, The Electronic Journal of Combinatorics, 15 , 1-13.

Chartrand, G. and Zhang, P., (2005), Introduction to graph Theory. McGraw-Hill Co., Inc.

Dinski, T. and Zhu, X., (1999), A Bound for the Game Chromatic Number of Graphs, Discrete Mathematics, 196, 109-115.

Faigle, U. et.al. (1993), On the Game Chromatic Number of Some Classes of Graphs, Ars Combinatorica, 35,143-150.

Guan, D.J. and X. Zhu, (1999), Game Chromatic Number of Outerplanar Graphs}, Journal of Graph Theory, 30, 67-70.

Kierstead, H.A., (1999), A Simple Competitive Graph Coloring Algorithm, Academic Press, 57-68.

Sia, Charmaine., (2009), The Game Chromatic Number of Some Families of Cartesian Product Graphs, (preprint)

Sidorowicz, E., (2007), The Game Chromatic Number and the Game Coloring Number of Cactuses, Information Processing Letters. 102, 147-151.

Zhu, X., (1999), Game Coloring Number of Planar Graphs, J. Combinatorial Theory Series B, 75, 245-258.