COOPERATION INDUCED BY SPONSORS

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Abstract

We focus on a game that involves two sets of players, $S = \{s_1; s_2; :::; s_m\}$ and $T = \{t_1; t_2; :::; t_n\}$. The members of S (referred to as sponsors) aim to induce cooperation among the members of T (called team players). Each member s_i offers a reward system v_i to the coalitions formed from T so that $v_i : 2^n \to R$ is a function giving the reward of a coalition $M \subseteq T$. On the other hand, the members of T may take either of the two strategies, to participate (0) or not (1) in any coalition M. The aggregate actions of members of S and T affect not only the rewards of the members of T but also of S who expect payoffs as well. A coalition then receives a total payoff of $\sum_{i=1}^{m} v_i(M)$ while a member s_i of S receives $G_i(M) - v_i(M)$ where $G_i(M)$ is the gross payoff to sponsor s_i once coalition M is formed. Just like in any cooperative game, the members of a coalition are concerned on how their group rewards are to be allocated "fairly." Hence, on the point of view of the players in T, allocations on $M^* = \max_{M \subseteq T} \sum_{i=1}^{m} v_i(M)$ is a major concern while each sponsor seeks to maximize his payoff from the result of M*.

Our focus in this study then is the formation of an equilibrium that is supposed to define an efficient outcome resulting from the strategies of the players both from S and T. We also seek to discuss some allocation strategies that will correspond to M^*.

Keywords: sponsors, coalitions, equilibrium, payoff

1 Introduction

Game theory, though relatively young as a mathematics field, has developed much because of its practical applications in real life situations. Two types of games the cooperative and non-cooperative cases, are usually the focus of discussions in this field. In both cases, solutions are sought after in the form of equilibria. This field of mathematics came about when John von Neumann published his paper entitled "On the Theory of Games of Strategy"in 1928. Research directed towards this field showed their utility in economics, political science, psychology, biology, and computer science and logic.

This paper focuses on establishing criteria that will induce an idea of equilibrium on what is referred to as games with sponsors which was introduced by the author in a previous paper [3]. Though the nature of the game is cooperative (seen from the perspective of the "team players"), it may be possible to view it too as non-cooperative.

2 Cooperative Games

Each cooperative game is associated with an ordered pair $\langle N, v\rangle$ consisting of the player set N and the characteristic function $v : 2^N \to R$ with $v(\emptyset) = 0$. Every nonempty subset of N is called a (crisp) coalition. Throughout our discussion, we will associate the game $\langle N, v\rangle$ to the characteristic function v. Hence, the real value $v(S)$ will represent the value that the members of set S will collect for cooperating. Also, it can be interpreted as the maximum potential gain of the players in S for cooperating.

The main problem of cooperative game theory is summed up with the question "if the grand coalition forms, how must the profit or cost savings $v(N)$ be divided among n players?"

Several solution concepts such as cores, Shapley value, the nucleolus and the like can be used to answer this question. Each solution concept determines how the cost savings obtained from the grand coalition should be distributed among the cooperating players while considering the potential rewards by examining all other different coalitions produced by the players. Therefore, every solution concept of a coalitional game is assigned to at least one payoff vector $x = (x_i)_{i \in N} \in R^n$, where player $i \in N$ receives the payoff x_i.

Definition 1 A set-valued solution (or a multi-solution) is a multifunction

$$F : G^N \to R^n$$

$$v \in G^N \to \text{a set of solutions}$$

Definition 2 A one-point solution (or a single-valued rule) is a map

$$f : G^N \to R^n$$

$$v \in G^N \mapsto (x_1, x_2, \ldots, x_n)$$

3 Sponsored Games $\langle S, T \rangle$

There are two sets players in this game, the sponsors S and the team players T. We use the notations $S = \{s_1, s_2, \ldots, s_m\}$ and $T = \{t_1, t_2, \ldots, t_n\}$ to denote these sets of players, respectively.

Each sponsor has a reward system that he offers to coalitions $M \subseteq T$ so that sponsor s_i's offer is effected by the payoff function $v_i : 2^{|T|} \to R$. Thus, for a coalition $M \subseteq T$, an n-tuple reward system may be associated and we denote this by $V(M) = (v_1(M), v_2(M), \ldots, v_m(M))$. On the other hand, each team player has the strategy set $\{1, 0\}$ denoting his joining or not joining a coalition M. Upon choosing to join M, he shares with the group's total reward

$$\sum V(M) = \sum_{i=1}^{m} v_i(M).$$

From hereon, we shall use the notation $\langle S, T \rangle$ to denote the sponsored game just described. In this game, a team player receives a reward for the action that he chooses based on the agreed allocation scheme that is computed from $\sum V(M)$. Just like in any common cooperative game solutions, it is desired that such allocation satisfies the individual rationality and efficiency properties. A team player t_j is concerned about how much payoff he will get from sponsors upon choosing to join a particular coalition. Team player t_j's payoff when he selects strategy 1 to join coalition M is $a_j(M)$ where

$$a(M) = (a_j)_{j \in M}$$

is an allocation associated with coalition M satisfying

(i) $a_j \geq \sum_{i=1}^{m} v(t_j)$ (individual rationality)

(ii) $\sum_{j \in M} a_j = \sum v(M)$ (efficiency)

Conversely, the sponsors are concerned with the actions of the team players and the payoff they will receive from that action. The payoff of sponsor s_i when he gives out the reward system v_i to coalition M is

$$b_i(M) = G_i(M) - v_i(M)$$

where $G_i(M)$ is the gross payoff to sponsor s_i when coalition M is formed. The sponsors may choose from their own strategy sets so that sponsor s_i may choose any reward system from the set S_i^v. A typical element of this strategy set for sponsor s_i may be viewed as a $2^{|T|}$-tuple denoted by $(v)^i_{2^{|T|}}$.

The extensive form of this game is described as follows. First, the sponsors each choose a reward system to offer to coalitions, all at the same time and without any cooperation with any other sponsor. The the team players, being informed of the rewards systems, choose a coalition to join. This means that the game consists of two stages: 1) the sponsors move simultaneously, and 2) the team players move simultaneously.

Situations that project this kind of game may be reflected in business places where sponsors could be thought of bosses who want to encourage team works that will benefit their company in terms of work efficiency and possibly profit gains. In the field of governance, politicians (congressmen or senators) may act as sponsors who intend to convince NGOs to act in unity to support some bill.

4 An Equilibrium for Sponsored Games

Definition 3 A pure-strategy equilibrium of a sponsored game hS; T i is a pair $(\hat{V}; \hat{M})$ where $\hat{V} = (v_i)_{s_i 2 S}$ and $\hat{M}$ is a coalition so that the following are satisfied.

(i) for every team player t_j and all reward systems of s_i 2 S, we have

$$a_j(\hat{M}) \; 2 \; \arg\max_{S_i^v} \left\{ (a_j(M))_{M \_ T} \right\}$$

(ii) for every sponsor s_i 2 S,

$$b_i(\hat{M}) \; 2 \; \arg\max_{M \_ T} \left\{ (b_i(M))_{S_i^v} \right\}$$

We desire a pure-strategy equilibrium of any sponsored game to satisfy the following three conditions.. These are derived from the idea that the sponsors choose their reward systems in order to give appropriate incentive to the team players.

A. Each team player makes a decision that will result into maximizing his payoff given the reward systems offered by all the sponsors.

B. If $(\hat{V}; \hat{M})$ is an equilibrium, then the cost of a deviation from $\hat{M}$ by team player t_j must be greater than the benefit of a deviation for all $M \_ T$ and s_i 2 S. There is no way that agent s_i would reduce his payoff corresponding to $\hat{M}$ without deviating from $\hat{V}$.

C. Given the reward systems offered by the sponsors, s_i can convince team players to join a particular coalition provided that he offers high enough payoff on that action. Each sponsor provides a payoff so that the cost of implementing a system V is minimal. The minimum cost for s_i to convince a team player t_j to deviate from any action to joining coalition M is given by $\sum_{j 2 M} \hat{v}_j(M)$.

From these conditions, a formal characterization is given in the following theorem.

Theorem 1 A pair $(\hat{V}; \hat{M})$ of reward system and coalition arises in a pure-strategy equilibrium if and only if the following conditions are satisfied:

(A) For every team member t_j 2 T and every reward system $(v)^i_{2 j T j}$ 2 S_i^v,

$$a_j(\hat{M}) \geq a_j(M): \tag{1}$$

(B) For each sponsor $s_i \in S$ and team player $t_j \in T$,

$$a_j(M) + b_i(\hat{M}) \geq a_j(\hat{M}) + b_i(M): \tag{2}$$

(C) For each team player $t_j \in T$ and every reward system $(v)^i_{\in T_j} \in S_i^v$,

$$\sum_{j \in M} \hat{v}_j(M) \geq \sum_{j \in M} v_j(M): \tag{3}$$

Proof.

Condition (A) requires that the allocation of team player t_j is such that for any reward system chosen by the members of S, we have

$$a_j(\hat{M}) = \max_{M \subseteq T; v \in S_i^v} a_j(M):$$

Thus, when (1) holds, this equation follows.

Requiring the cost of deviation from $\hat{M}$ by team player t_j must be greater than the benefit of a deviation for all $M \subseteq T$ and $s_i \in S$ means

$$a_j(M) \quad a_j(\hat{M}) \geq b_i(M) \quad b_i(\hat{M}) \tag{4}$$

and this is basically what is stated in (2).

From the point of view of a sponsor s_i, the cost of his support for the members of any coalition M must be minimized so as to maximize his own payoff. Therefore, (3) must hold.

Corollary 1.1 An equilibrium pair $(\hat{V}; \hat{M})$ of a sponsored game $\langle S; T \rangle$ satisfies each of the following:

$$\sum_{j \in M} \hat{v}_j(M) = \min_{v \in S_i^v; s_i \in S} \sum_{j \in M} v_j(M): \tag{5}$$

$$b_i(\hat{M}) = \max_{M \subseteq T; v \in S_i^v} b_i M \tag{6}$$

.

Proof.

These are but the requirements of minimizing the cost and maximizing the gain of each sponsor $s_i \in S$.

5 Conclusion and Recommendation

This paper characterizes a pure-strategy equilibrium of a sponsored game $\langle S; T \rangle$. The focus is on identifying conditions that enable the two sets of players to choose strategies that will make them gain the best payoff with each team player wanting to maximize his allocation by joining the best coalition that gives him his best payoff and with each sponsor minimizing his cost in convincing the team players to join his chosen coalition and at the same time and in return maximizing his gain from such action.

This type of game is still rich for further studies that will focus on any of the following:

1. designing an allocation given the specified reward systems of all the sponsors
2. characterization for specific cases such as minimizing the strategy sets of the members S and T
3. extending the corresponding crisp cooperative nature of the strategies of T to focus on fuzzy coalitions

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