THE NORMAL FORM OF A GRAPH COLORING PROBLEM ON PATHS

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Abstract

The graph coloring game described in this paper has two players, Alice and Bob. Both of them take turns in coloring vertices of a graph, with the only rule that adjacent vertices cannot have the same color. Alice wins if all vertices have been colored. Otherwise, Bob wins.

The normal form of a game includes all strategies and the corresponding payoffs of each player. When there are only two players, the normal form of game can be represented by a matrix.

The objective of this paper is to represent the graph coloring game played by Alice and Bob on paths in its normal form.

Keywords: graph coloring game; game chromatic number; path; normal form, payoff

1 Introduction

One particular focus on graph coloring is a game on coloring the vertices of the graph properly. A proper coloring means to color adjacent vertices with different colors. This paper presents the normal form of such a game on paths.

Definition 1. A path P_n is a non-empty graph, with vertex set $V(P_n) = \{v_1, v_2, \ldots v_n\}$ and edge set $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$.

Given a path with at least three vertices, we define the graph coloring game played by two players, Alice and Bob, as a game where the payoff each player receives depend on whether Alice wins or not.

Definition 2. Let N denote the set of positive integers and 2^N denote the power set of N. Let $G = (V, E)$ be a graph. A function $f : V \to N$ is a colouring of G if $f(v) \neq f(u)$ whenever $vu \in E$.

We consider the two-player game between Alice and Bob. They play alternatively with Alice having the first move. Given a graph G and a set C of colours, the players take turns colouring G with colours from C. If after $|V(G)|$ moves the graph is coloured then Alice wins, otherwise Bob wins.

Definition 3. The game chromatic number of G, denoted by $\chi_g(G)$, is defined as the least cardinality of C for which Alice has a winning strategy.

Theorem 1. A path of order n has game chromatic number

$$\chi_g(P_n) = \begin{cases} & \text{if } 2 \le n \le 3, \\ 3, & \text{if } n \ge 4. \end{cases}$$

Proof. Let P_n be a path of order n with vertex set $V(P_n) =$ f$v_1; v_2; :::v_n$g and edge set $E(P_n) =$ f$v_1v_2; v_2v_3; :::; v_{n\ 1}v_n$g.

It is known that the chromatic number of a graph is a lower bound for its game chromatic number. Hence, it is obvious that $\_{g}(P_2) = 2$. If the order of the path is 3, Alice's best move is to always color the middle vertex, v_2 first. This will assure that all the vertices will be properly colored with only 2 colors.

For a path of order at least 4, 2 colors will not be enough to guarantee a victory for Alice. After Alice colors a vertex, the best strategy for Bob is to color a vertex of distance 2 with a different color. Since there are at least 4 vertices, we can see that Alice will be forced to color the vertex adjacent to the vertex she first chose with a color different from the two vertices already colored. The remaining uncolored vertices may already be colored using the three colors used depending on adjacency.

□

The normal form, also known as the matrix form, is the most familiar representation of strategic interactions in game theory.

Definition 4. A (finite, n-person) normal-form game is a tuple $(N; A; u)$, where:

- _ N is a finite set of n players, indexed by i;
- _ $A = A_1$ _ $:::$ _ A_n, where A_i is a finite set of actions available to player i. Each vector $a = (a_1; :::; a_n)$ 2 A is called an action profile;
- _ $u = (u_1; :::; u_n)$ where $u_i : A$! R is a real-valued payoff function for player i.

Example 1. In the game of matching coins, player 1 chooses "heads" (H) or "tails"(T). Player 2 not knowing player 1's choice, also chooses "heads" or "tails". If two choose alike, then player 2 wins a cent from player 1; otherwise, player 1 wins a cent from 2. A two-player game like this in normal form can be represented as a matrix. The rows of the matrix represent the action set of player 1 while the columns of the matrix represents the actions set of player 2. The cell indexed by row x and column y contains a pair, $(a; b)$, where a is the payoff to player 1 and b is the payoff to player 2, that is, $a = u_1(x; y)$ and $b = u_2(x; y)$. Figure 1 illustrates this.

	H	T
H	(1; 1)	(1; 1)
T	(1; 1)	(1; 1)

Figure 1: Matching Coin Game

To better understand the normal form of the game presented in this paper, a a game tree also known as the extensive form will first be presented. We will consider the following payoff; if Alice wins, Bob pays her a cent. Otherwise, she pays him a cent.

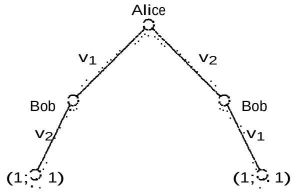

Figure 2: Extensive Form on P_2

	M
L	(1; 1)
R	(1; 1)

Figure 3: Normal Form on P_2

2 The Normal Form on P_2

A path of order 2 has only two vertices. No matter which vertex Alice choose to start from, she is guaranteed to win according to the rules of the game. The game tree on P_2 is shown in figure 2.

There are only two choices for Alice when she makes her first move, we will represent her moves as L for left and R for right. Bob has no choice but to color the other vertex with a different color, we will denote his move by M. Hence, the payoff is the same at each terminal vertex. Bob has to pay Alice a cent. This shows that the game chromatic number of P_2 is 2.

The normal form of this game is shown in figure 3 where Alice is the row player and Bob is the column player.

3 The Normal Form on P_3

Let P_3 be the path with vertex set f$v_1; v_2; v_3$g and edge set f$v_1v_2; v_2v_3$g.

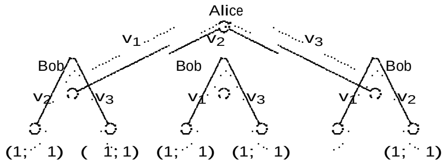

Figure 4: Extensive Form on P_3

From the game tree in figure 4, if Alice choose v_2, she is guaranteed to win. There are only two colors needed to properly color the graph. However, if she chooses either v_1 or v_3, Bob will surely win since he will always choose to color the vertex of distance 2 from the first vertex Alice colored. In this case, there must be three colors to color all the vertices properly.

Alice always makes the first move. It will make no sense if she will choose a vertex other than v_2. This shows the game chromatic number of P_3 is 2.

We represent the moves of Alice as L for Left, M for Middle and R for Right. For each move of Alice, Bob has the choice of either to go left or right. Each of his actions in response to the move of Alice must be specified. We will represent his moves as xyz, where x; y and z are each either L or R. Thus the actions of Bob are LLL; LLR; LRL; LRR; RRR; RRL; RLR; RLL. The normal form of the game is shown in figure 5.

	LLL	LLR	LRL	LRR	RRR	RRL	RLR	RLL
L	(1; 1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)
M	(1; 1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)
R	(-1,1)	(-1,1)	(-1,1)	(-1,1)	(1; 1)	(1; 1)	(1; 1)	(1; 1)

Figure 5: Normal Form on P_3

4 The Normal Form on P_4

Let P_4 be the path with vertex set fv_1; v_2; v_3; v_4g and edge set fv_1v_2; v_2v_3; v_3v_4g.

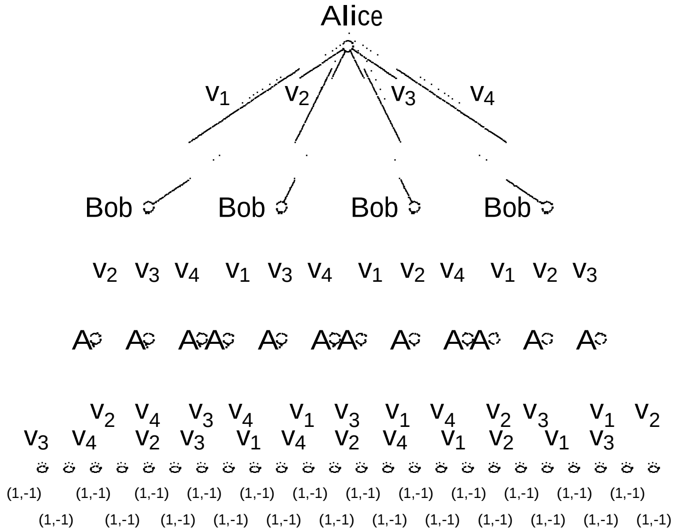

Figure 6: Extensive Form on P_4

The extensive form of the game on P_4 is shown in figure 6. From the proof of Theorem 1, after Alice colors a vertex, the best strategy for Bob is to color a vertex of distance 2 with a different color. This means that if Alice chose v_1, Bob will not choose to color v_2 for this guarantees a win for Alice. If this will be the case, and Alice will choose to color v_4 with the same color Bob used, 2 colors will be enough to properly color the graph. If she chose the same color she used, 3 colors are needed. In both cases, she wins.

Now let us consider the best strategy for Bob. He will obviously choose to color v_3 with a different color if Alice chose v_1. A third color will be used by Alice to color v_2 and she is still guaranteed to win. If Bob chose to color vertex v_4 after Alice chose v_1. Whatever color he chooses for this vertex will still guarantee a win for Alice.

Suppose Alice chose to color v_2 first. If Bob chose to color a vertex adjacent to this one, Alice wins. If he chose to color v_4, only three colors are needed and Alice will still win. The case of Alice choosing to color v_3 on her first move is similar. This is because both vertices are of degree 2 and both are adjacent to a pendant. Similarly, the case for v_4 is the same as that of v_1 since both are pendant vertices.

The payoff at the terminal vertices are the same. This means that given three colors, Alice is assured of winning.

To represent this game in normal form, we will represent the actions of Alice as $Lxyz; Mxyz; Nxyz$ and $Rxyz$ where $L; M; N$ and R corresponds to Alice's first move while xyz will correspond to her move in response to Bob's move. Each of $x; y$ and z corresponds to either left L or right R. She has 4×8 moves. On the other hand, Bob's move will be represented by abcd in response to Alice's first move. He has 3^4 moves. The set of actions A_1 of Alice is given as

$$A_1 = \{LLLL; LLLR; LLRL; LRLL; LLRR; LRLR; LRRL; LRRR; MLLL; MLLR; MLRL; MRLL; MLRR; MRLR; MRRL; MRRR; NLLL; NLLR; NLRL; NRLL; NLRR; NRLR; NRRL; NRRR; RLLL; RLLR; RLRL; RRLL; RLRR; RRLR; RRRL; RRRR\}$$

while a partial list of the set of actions A_2 of Bob is given as

$$A_2 = \{LLLL; LLLR; LLRL; LRLL; RLLL; LLLM; LLML; LMLL; MLLL; LLMR; LLRM; LMRL; LRML; RMLL; MRLL; MLLR; RLLM; \ldots\}$$

Thus, the normal form of this game is a 32×81 matrix. We will only show in figure 7 the first nine moves of Bob. The matrix for the rest of his moves is similar.

	LLLL	LLLR	LLRL	LRLL	RLLL	LLLM	LLML	LMLL	MLLL
LLLL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
LLLR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
LLRL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
LRLL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
LLRR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
LRLR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
LRRL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
LRRR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
MLLL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
MLLR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
MLRL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
MRLL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
MLRR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
MRLR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
MRRL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
MRRR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
NLLL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
NLLR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
NLRL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
NRLL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
NLRR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
NRLR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
NRRL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
NRRR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
RLLL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
RLLR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
RLRL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
RRLL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
RLRR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
RRLR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
RRRL	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
RRRR	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)

Figure 7: Normal Form on P_4

5 The Normal Form on $P_n, n > 4$

As with the previous discussions, let P_n be the path with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1v_2, \ldots, v_{n-1}v_n\}$. As a particular example, we consider P_5. Given this path, Alice will have five choices for her first move.

From Theorem 1, the game chromatic number of P_n is 3 if n is greater than 4. This means that no matter which vertex Alice chose for her first move, Bob needs to pay him a cent if there are three colors available. Hence, the payoff for all terminal vertices of its game tree must be $(1, -1)$.

To illustrate, suppose Alice choose a vertex of degree one of the path P_5 for her first move. In particular, suppose v_1 was her choice. The best move for Bob is to choose to color a vertex of distance 2, v_3, and color it with a different one. If Alice will choose v_2 as her next move she has no choice but to color it with the third color. The next two vertices can be colored properly as they take turns to finish the game.

If Alice choose a vertex of degree two for her first move, say v_2 and Bob chooses to color v_4, all the remaining vertices can be colored properly. Here, it is clear that the minimum number of colors which assures Alice of winning is three. A similar illustration may be given for any path of order 6 or more.

Now, for the normal form of the game on $P_n, n > 4$, we will only describe how to enumerate the actions of Alice and Bob. Let us consider P_5. Since this is a sequential game, Alice has five choices in her first move we may represent her moves as Awxyz where A corresponds to the first vertex that she will choose while wxyz will correspond to her moves in response to Bob's move.

For Bob, his actions may be represented as abcdetuv where abcd is his response to Alice's first move and tuv is his response to her second move. The normal form is similar to figure 7, only with more rows and columns. For P_5, there are 16×5 rows and $4^5 \times 3^4$ columns.

We note here that this is a game of perfect information. Alice and Bob would choose the best move when they take their turn. And we have shown that if $n > 4$ a minimum of three colors will be needed to color the path P_n for Alice to win.

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