DYNAMICS OF GALACTIC DISKS UNDER LINEARIZED GRAVITATION

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Abstract: We considered in this paper a general relativistic treatment of dynamics in galaxies. In particular, we studied the first-order correction to Newtonian gravity by using linearized gravitation to determine the equation of motion of star systems in galaxies. This is carried out by first finding out how first-order deviation from flat spacetime which is generated by the potential of the galactic disk. The resulting metric was then used to determine the Christoffel symbols, which when included in the geodesic equation yielded the equation of motion.

1. INTRODUCTION

The velocity curves of some galaxies, which at some distance from galactic centers, were shown to no longer vary with distance are peculiar because they are contrary to the predictions using Newtonian mechanics and Newtonian gravitation, that speed be inversely proportional to the distance.

Solutions suggested varied from dark matter (Faber & Gallagher, 1979), modification of Newtonian dynamics (Bekenstein J. D., 2007), or non-Newtonian gravity (see for example, Van Nieuwenhove, 2007). It should be noted that galactic dynamics are typically analyzed using Newtonian gravity and Newtonian dynamics. Dark matter theories essentially accept the validity of both Newtonian gravitation and dynamics but postulate the existence of unseen matter which by some calculations account for 25% of the energy in the Universe (Faber & Gallagher, 1979).

Since the advent of Einstein's General Theory of Relativity, it has been generally accepted that Newtonian gravity is but the weak-field limit of Einstein's theory. Standard theory of galactic dynamics are however to this day generally Newtonian in orientation. Recently, there have been attempts to fit the observed velocity curves within the framework of Einstein's theory. The objectives of these studies had been to find the potentials that would be consistent with the observed velocity curves (Cooperstock & Tieu, 2006).

We applied in this study the converse approach. Using the potentials known to apply to galaxies, particularly that of the galactic disk, we considered a linearized gravity approximation to General relativity to obtain the equations of motion of the galaxy.

2. LINEARIZED GRAVITY

In the weak field limit, gravity can be treated as a perturbation theory where

$$g_{ab} = \eta_{ab} + \gamma_{ab} \qquad \textbf{(Eq. 1)}$$

where g is the metric of the spacetime manifold, η is the Minkowski metric, and γ the perturbation. The Einstein tensor may then be cast as

$$G^{(1)}_{ab} = \partial^c \partial_{(b}\gamma_{a)c} - \frac{1}{2}\partial^c\partial_c\gamma_{ab} - \partial_a\partial_b\gamma - \frac{1}{2}\eta_{ab}\left(\partial^c\partial^d\gamma_{cd} - \partial^c\partial_c\gamma\right) \qquad \textbf{(Eq. 2)}$$

With a gauge transformation

$$\bar{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma \qquad \textbf{(Eq. 3)}$$

the linearized Einstein equation becomes

$$\partial^c\partial_c\bar{\gamma}_{ab} = -16\pi T_{ab} \qquad \textbf{(Eq. 4)}$$

The problem boils down to one of determining the perturbation γ. For this study, we assume that just beyond the Newtonian limit, the gravitational potential Φ still satisfies the Poisson's equation

$$\nabla^2\bar{\gamma}_{00} := \nabla^2\Phi = -16\rho \qquad \textbf{(Eq. 5)}$$

Under this condition, the perturbation γ may be obtained using (Wald, 1984)

$$\gamma_{ab} = \bar{\gamma}_{ab} - \frac{1}{2}\eta_{ab}\gamma = -\left(4t_a t_b + 2\eta_{ab}t^c t_c\right)\Phi \qquad \textbf{(Eq. 6)}$$

3. DYNAMICS OF GALACTIC DISKS

Dynamics on galactic disks are generally described by the potential (Binney & Tremaine, 2008)

$$\Phi_L = \frac{1}{2}{v_0}^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{{q_\Phi}^2}\right) + \text{constant} \qquad \textbf{(Eq. 7)}$$

where R_c and v_0 are constants, and q_Φ is the axis ratio of the equipotential surfaces which controls the flattening of these surfaces. This potential satisfies the Poisson's equation. For such a potential, it is convenient to work with cylindrical coordinates, thus with the Minkowski metric η = diag (-1, 1, R^2, 1), Eqs. 6 and 7 yield the following perturbation terms:

$$\gamma_{tt} = \frac{2{v_0}^2}{\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right)^2}\left[\left(R_c^2 - R^2 + \frac{z^2}{q_\Phi^2}\right) + \frac{1}{q_\Phi^2}\left(R_c^2 + R^2 - \frac{z^2}{q_\Phi^2}\right)\right] \quad \textbf{(Eq. 8)}$$

$$\gamma_{\theta\theta} = \frac{-2{v_0}^2 R^2}{\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right)^2}\left[\left(R_c^2 - R^2 + \frac{z^2}{q_\Phi^2}\right) + \frac{1}{q_\Phi^2}\left(R_c^2 + R^2 - \frac{z^2}{q_\Phi^2}\right)\right] \quad \textbf{(Eq. 9)}$$

$$\gamma_{zz} = \frac{-{v_0}^2}{\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right)^2}\left[2\left(R_c^2 - R^2 + \frac{z^2}{q_\Phi^2}\right) + \frac{6}{q_\Phi^2}\left(R_c^2 + R^2 - \frac{z^2}{q_\Phi^2}\right)\right] \quad \textbf{(Eq. 10)}$$

$$\gamma_{tz} = \gamma_{zt} = \frac{8{v_0}^2 Rz}{q^2\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right)^2} \quad \textbf{(Eq. 11)}$$

All other γ are zero.

The equations of motion may be obtained from the geodesic equations

$$\frac{\partial^2 x^\mu}{\partial\tau^2} + \sum_{\rho\sigma}\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \quad \textbf{(Eq. 12)}$$

where Γ are the christoffel symbols, which in the linearized gravity limit are

$$\Gamma^c_{ab} = \frac{1}{2}\eta^{cd}\left(\partial_a\gamma_{bd} + \partial_b\gamma_{ad} - \partial_c\gamma_{ab}\right) \quad \textbf{(Eq. 13)}$$

Applying Eqs. 8 – 11 and Eq. 13 to Eq. 12, we obtain the equations of motion

$$\frac{\partial^2 t}{\partial s^2} +$$

$$2\left\{\frac{2{v_0}^2 R}{\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(3 + \frac{1}{q_\Phi^2}\right) - R^2\left(1 - \frac{1}{q_\Phi^2}\right) + \frac{3z^2}{q_\Phi^2}\left(1 + \frac{1}{q_\Phi^2}\right)\right]\right\}\frac{dt}{ds}\frac{dR}{ds} + \quad \textbf{(Eq.14)}$$

$$2\left\{\frac{2{v_0}^2 z}{q^2\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(1 + \frac{3}{q_\Phi^2}\right) - \left(3R^2 - \frac{z^2}{q_\Phi^2}\right)\left(1 - \frac{1}{q_\Phi^2}\right)\right]\right\}\frac{dt}{ds}\frac{dz}{ds} = 0$$

$$
\begin{aligned}
&\frac{\partial^2 R}{\partial s^2}+\\
&\left\{\frac{-2{v_0}^2 R}{\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(3+\frac{1}{q_\Phi^2}\right)-R^2\left(1-\frac{1}{q_\Phi^2}\right)+\frac{3z^2}{q_\Phi^2}\left(1+\frac{1}{q_\Phi^2}\right)\right]\right\}\frac{dt}{ds}\frac{dt}{ds}+\\
&\left\{\frac{-4{v_0}^2 R}{\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(6+\frac{1}{q_\Phi^2}\right)-R^2\left(3-\frac{1}{q_\Phi^2}\right)+\frac{z^2}{q_\Phi^2}\left(6+\frac{3}{q_\Phi^2}\right)\right]\right\}\frac{dR}{ds}\frac{dR}{ds}+\\
&\left\{\frac{2{v_0}^2 R}{\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[\begin{array}{l}R_c^2\left(1+\frac{1}{q_\Phi^2}\right)\left(R_c^2-R^2+\frac{z^2}{q_\Phi^2}\right)-\left(1-\frac{1}{q_\Phi^2}\right)\\ \left\{\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)\left(2R^2-\frac{z^2}{q_\Phi^2}\right)+2R^2\left(R^2-\frac{z^2}{q_\Phi^2}\right)\right\}\end{array}\right]\right\}\frac{d\theta}{ds}\frac{d\theta}{ds}+\\
&\left\{\frac{2{v_0}^2 R}{\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[\left(R_c^2+\frac{z^2}{q_\Phi^2}\right)\left(3+\frac{3}{q_\Phi^2}\right)-R^2\left(1-\frac{3}{q_\Phi^2}\right)\right]\right\}\frac{dz}{ds}\frac{dz}{ds}+\\
&2\left\{\frac{-4{v_0}^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2-3R^2+\frac{z^2}{q_\Phi^2}\right]\right\}\frac{dt}{ds}\frac{dz}{ds}+\\
&2\left\{\frac{-4{v_0}^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(1+\frac{1}{q_\Phi^2}\right)-\left(3R^2-\frac{z^2}{q_\Phi^2}\right)\left(3-\frac{1}{q_\Phi^2}\right)\right]\right\}\frac{dR}{ds}\frac{dz}{ds}=0
\end{aligned}
$$

(Eq.15)

$$\frac{\partial^2 z}{\partial s^2}+$$

$$\left\{\frac{2{v_0}^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(1+\frac{3}{q_\Phi^2}\right)-\left(3R^2-\frac{z^2}{q_\Phi^2}\right)\left(1-\frac{1}{q_\Phi^2}\right)\right]\right\}\frac{dt}{ds}\frac{dt}{ds}+$$

$$\left\{\frac{-2{v_0}^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[\left(R_c^2+\frac{z^2}{q_\Phi^2}\right)\left(3+\frac{1}{q_\Phi^2}\right)-3R^2\left(3-\frac{1}{q_\Phi^2}\right)\right]\right\}\frac{dR}{ds}\frac{dR}{ds}+$$

$$\left\{\frac{-2{v_0}^2 R^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(1+\frac{3}{q_\Phi^2}\right)-\left(3R^2-\frac{z^2}{q_\Phi^2}\right)\left(1-\frac{1}{q_\Phi^2}\right)\right]\right\}\frac{d\theta}{ds}\frac{d\theta}{ds}+$$

$$\left\{\frac{2{v_0}^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(1+\frac{9}{q_\Phi^2}\right)-3R^2\left(1-\frac{3}{q_\Phi^2}\right)+\frac{3z^2}{q_\Phi^2}\left(1+\frac{1}{q_\Phi^2}\right)\right]\right\}\frac{dz}{ds}\frac{dz}{ds}+$$

$$2\left\{\frac{4{v_0}^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2-3R^2+\frac{z^2}{q_\Phi^2}\right]\right\}\frac{dt}{ds}\frac{dR}{ds}+$$

$$2\left\{\frac{2{v_0}^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[3R_c^2\left(1+\frac{1}{q_\Phi^2}\right)-\left(R^2-\frac{3z^2}{q_\Phi^2}\right)\left(1-\frac{3}{q_\Phi^2}\right)\right]\right\}\frac{dR}{ds}\frac{dz}{ds}=0$$

(Eq.16)

$$\frac{\partial^2\theta}{\partial s^2}+ 2\left\{\frac{2{v_0}^2 R}{\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(3+\frac{1}{q_\Phi^2}\right)-R^2\left(1-\frac{1}{q_\Phi^2}\right)+\frac{z^2}{q_\Phi^2}\left(1-\frac{3}{q_\Phi^2}\right)\right]\right\}\frac{dR}{ds}\frac{d\theta}{ds}+ 2\left\{\frac{2{v_0}^2 z}{q_\Phi^2\left(R_c^2+R^2+\frac{z^2}{q_\Phi^2}\right)^3}\left[R_c^2\left(1+\frac{3}{q_\Phi^2}\right)-\left(3R^2-\frac{z^2}{q_\Phi^2}\right)\left(1-\frac{1}{q_\Phi^2}\right)\right]\right\}\frac{d\theta}{ds}\frac{dz}{ds}=0 \qquad \textbf{(Eq.17)}$$

4. DISCUSSIONS

We have determined in this paper the geodesic equations using linearized Einstein's equation. Preliminary results show that when the logarithmic potential (Eq. 7) of the galactic disk was incorporated in the perturbation term γ, the effect on the galactic dynamics is complex. Evidently, the geodesic equations (Eq.14 to Eq.17) are coupled nonlinear differential equations. Further calculations may be done to solve for the velocity curves of the disk galaxy from these geodesic equations. Then the results can be compared to what is expected in Newtonian theory.

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