GRAVITATION AS GAUGE THEORY OF THE POINCARÉ GROUP

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Abstract: A challenge faced in the quantization of the gravitational field, and its possible unification with other interactions is that gravitation is described differently from the other three fundamental forces – weak, strong, and electromagnetic. While Einstein's gravitational theory is a geometric theory, the standard model of particle interactions which describes the other three forces is field theoretic. In this paper, we showed that a field theoretic gravitational theory can be developed by considering a gauge theory of the Poincaré group. Symmetry under a local Poincaré transformation requires the introduction of six rotational gauge fields and four translational gauge fields. We then showed that application of the minimum action principle to a simple free-field Lagrangian can lead to the Einstein's field equations, with the translational fields being related to the energy-momentum tensor. The existence of the rotational gauge fields however leads to torsion, so the resulting gravitational theory is potentially more general than Einstein's theory. Since torsion is related to spinning matter, and spin averages out on a sufficiently large scale, the torsion-free Einstein theory could be considered as the spin-free limit of the gauge theory. Quantization of the field on the other can proceed in a manner similar to those of other interactions.

Keywords: gravitation; general relativity; gauge theory; Poincaré group; torsion

1. INTRODUCTION

Unification of the fundamental forces of nature has so far been successful only with the electromagnetic, strong, and weak interactions. The main difficulty that one encounters in trying to unify gravitational forces with the other three is that traditionally, gravitation has been a geometric theory, whereas the other three are field theories. It is the objective of this paper then, to bridge this gap in language, by resenting a gauge-theoretic approach to gravitation. The form of interaction between well-known fields, such as electromagnetic and Yang-Mills fields can be determined by postulating invariance of the action under a group of transformation. It will be shown in this paper that under a Poincaré transformation, we are led to a theory of gravitation. One significant difference of the resulting theory with Einstein's theory is the presence of torsion. It will be shown that torsion is associated with the spin of matter. Spin is expected to average out in the large, so we expect this theory to reduce to a torsion-free model, just as Einstein's theory is. Einstein's theory has been successfully tested in the macroscopic scale, but it is not taken to be valid in the microscopic scale. While some approaches to quantum gravity have shown promise, it is today still an open field. It is largely believed that unification of gravity with the other forces will manifest itself in a successful theory of quantum gravity.

2. GAUGE THEORY

Systems of particles may be described by Lagrangians $L = T - V$ where T is the kinetic energy and V the potential energy. The Lagrangian is taken to be a function of generalized coordinates q and velocities $\dot{q}$. By Hamilton's principle, the motion of a classical particle is obtained when the action $S = \int L(q,\dot{q})\,dt$ is minimized. This principle of least action leads to the equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad \textbf{(Eq. 1)}$$

If the system is symmetric under some transformation, say $q \rightarrow q' = Aq$, where A is the transformation operator, the Lagrangian remains invariant under the transformation. This consequently leads to a conservation law

$$\frac{dJ}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) = 0 \qquad \textbf{(Eq. 2)}$$

Fields may likewise be described by a Lagrangian density L from which minimization of the action $S = \int d^4x\,\mathscr{L}(\psi, \partial_\mu \psi)$ leads to the field equations

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)}\right) - \frac{\partial \mathscr{L}}{\partial \psi} = 0 \qquad \textbf{(Eq. 3)}$$

If the system is symmetric under a global transformation $\delta\psi^A = \varepsilon^a T^A_{aB}\psi^B$,where ε is a constant parameter, and T are the generator of the Lie group $[T_a, T_b]^A_B = f^c_{ab}T^A_{cB}$, the Lagarangian is invariant, and we have the conservation law $J^\mu_{a,\mu} \equiv \partial_\mu J^\mu_a = 0$ with the conserved current being

$$J^\mu_a = -\left(\frac{\delta \mathscr{L}}{\delta \partial \psi^A_{,\mu}} T^A_{aB}\psi^B\right) \qquad \textbf{(Eq. 4)}$$

Under a local transformation where the parameter ε varies with position, $\varepsilon^a = \varepsilon^a(x)$,

$$\delta \mathscr{L} = \frac{\delta \mathscr{L}}{\delta \psi^A_{,\mu}} T^A_{aB}\psi^B \varepsilon^a_{,\mu} \qquad \textbf{(Eq. 5)}$$

For the Lagrangian to remain invariant, a new field B^J has to be introduced to compensate for the term on the right side of Eq. 5. Hence, we must now recast the Lagrangian as $\mathscr{L} = \mathscr{L}(\psi; \psi_{,\mu}; A^J)$. This Lagrangian is invariant under the local transformation $\delta\psi^A = \varepsilon^a(x) T^A_{aB}\psi^B$, provided that the gauge field transforms as $\delta B^J = U^J_{aK}B^K\varepsilon^a(x) + C^{J\mu}_a \varepsilon^a_{,\mu}$. For then, the following relations can be ensured for arbitrary ε,

$$\frac{\delta \mathscr{L}}{\delta \psi^A} T^A_{aB}\psi^B + \frac{\delta \mathscr{L}}{\delta \psi^A_{,\mu}} T^A_{aB}\psi^B_{,\mu} + \frac{\delta \mathscr{L}}{\delta B^J} U^J_{aK}B^K = 0 \qquad \textbf{(Eq. 6)}$$

$$\frac{\delta \mathscr{L}}{\delta \psi^A_{,\mu}} T^A_{aB}\psi^B + \frac{\delta \mathscr{L}}{\delta B^J} C^{J\mu}_a = 0 \qquad \textbf{(Eq. 7)}$$

The last equation implies that A and $\partial_\mu\psi$ may appear in the Lagrangian only through the covariant derivative $D_\mu\psi^A \equiv \psi^A_{,\mu} - T^A_{aB}\psi^B A^a_\mu$, where we redefined the gauge field as $A^a_\mu = \left(C^{-1}\right)^a_{\mu J} B^J$. Eq. 6 may then be recast as

$$\frac{\delta\mathcal{L}}{\delta\psi^A}T^A_{aB}\psi^B + \frac{\delta\mathcal{L}}{\delta D_\mu\psi^A}T^A_{aB}D_\mu\psi^B + \frac{\delta\mathcal{L}}{\delta D_\mu\psi^A}\psi^B A^b_\nu\left\{\left[T_a,T_b\right]^A_B\delta^\nu_\mu - S^{d\nu}_{ab\mu}T^A_{dB}\right\} = 0 \qquad \textbf{(Eq. 8)}$$

where $S^{a\nu}_{cb\mu} = \left(C^{-1}\right)^a_{\mu J} U^J_{cK} C^{K\nu}_b$. We then note that $\delta D_\mu\psi^A = T^A_{aB}D_\mu\psi^B\varepsilon^a(x)$, and $\delta A^a_\mu = f^a_{cb}A^b_\mu\varepsilon^c(x) + \varepsilon^a_{,\mu}$. Thus, a system cannot retain its symmetry under a local transformation unless we consider it as part of a larger system. This means that the system can no longer be treated in isolation and interaction must be brought into the picture. With the introduction of a gauge field, it is imperative that we also include the free Lagrangian of this field: $\mathcal{L}_0\left(A^a_\mu; A^a_{\mu,\nu}\right)$. To preserve the gauge invariance of the overall Lagrangian, this "free-field" Lagrangian must also be invariant under the transformation. Taking the coefficients of ε, $\varepsilon_{,\mu}$, and $\varepsilon_{,\mu\nu}$, in δL separately, we have

$$\frac{\delta\mathcal{L}_0}{\delta A^a_\mu}f^a_{bc}A^c_\mu + \frac{\delta\mathcal{L}_0}{\delta A^a_{\mu,\nu}}f^a_{bc}A^c_{\mu,\nu} = 0 \qquad \textbf{(Eq. 9)}$$

$$\frac{\delta\mathcal{L}_0}{\delta A^a_\mu} + \frac{\delta\mathcal{L}_0}{\delta A^a_{\nu,\mu}}f^a_{bc}A^c_\nu = 0 \qquad \textbf{(Eq. 10)}$$

$$\frac{\delta\mathcal{L}_0}{\delta A^a_{\mu,\nu}} + \frac{\delta\mathcal{L}_0}{\delta A^a_{\nu,\mu}} = 0 \qquad \textbf{(Eq. 11)}$$

The last equation indicate that $A_{\mu,\nu}$ enters the Lagrangian only through $A^a_{[\mu,\nu]} \equiv A^a_{\mu,\nu} - A^a_{\nu,\mu}$. Eq. 10 thus gives

$$\frac{\delta\mathcal{L}_0}{\delta A^a_\mu} = \frac{\delta\mathcal{L}_0}{\delta A^a_{[\mu,\nu]}}f^a_{bc}A^c_\nu \qquad \textbf{(Eq. 12)}$$

which in turn suggests that A_μ and $A_{\mu,\nu}$ appears in the Lagrangian only through $F^a_{\mu\nu} = A^a_{[\mu,\nu]} - \frac{1}{2}f^a_{bc}\left(A^b_\mu A^c_\nu - A^b_\nu A^c_\mu\right)$. This further implies that the Langrangian is a function of F alone: $\mathcal{L}_0\left(A^a_\mu; A^a_{\mu,\nu}\right) = \mathcal{L}_0\left(F^a_{\mu\nu}\right)$. Applying to Eq. 9 then gives

$$\frac{1}{2}\frac{\delta\mathcal{L}_0}{\delta F^a_{\mu\nu}}f^a_{bc}F^b_{\mu\nu} = 0 \qquad \textbf{(Eq. 13)}$$

We also note that $\delta F^a_{\mu\nu} = \varepsilon^b(x)f^a_{bc}F^c_{\mu\nu}$.

3. GAUGE THEORY OF THE POINCARÉ GROUP

Let us consider the transformation $x^i \rightarrow \pi x^i := x^i + \tilde{\omega}^i_j x^j + \tilde{\varepsilon}^i$, and $\psi(x) \rightarrow [\pi\psi](x) := \left(1 + \omega^{ab} f_{ba} - \varepsilon^a \partial_a\right)\psi(x)$. We assume the spacetime to be locally Minkowskian and denote the local variables by greek indices. Latin indices refer to the world frame. ω are rotation parameters while ε are translation parameters $\varepsilon^a := \tilde{\varepsilon}^a + \tilde{\omega}^a_b \delta^b_i x^i$, and $\omega^{ab} = \tilde{\omega}^{ab}$. Suppose the generators of rotation $f_{\alpha\beta}$, and that of translation $\partial_\alpha = \delta_\alpha{}^i \partial_i$, satisfy the algebra of the Poincaré group (a) $\left[f_{ab}, f_{gd}\right] = g_{g[a} f_{b]d} - g_{d[a} f_{b]g}$, (b) $\left[f_{ab}, \partial_g\right] = g_{g[a} \partial_{b]}$, and (c) $\left[\partial_a, \partial_b\right] = 0$, where g are the metric tensor. Since Poincaré transformation is a spacetime transformation, we require not only the invariance of just the Lagrangian but that of the action instead

$$\delta S = 0 = \pi S - S = \int_{\pi\Omega} \mathscr{L}\left(\pi\psi, \partial_i \pi\psi\right) d^4x - \int_\Omega \mathscr{L}\left(\psi, \partial_i\psi\right) d^4x$$

$$= \int_\Omega d^4x \left\{ \left(\frac{\delta\mathscr{L}}{\delta\psi} - \partial_i \frac{\delta\mathscr{L}}{\delta\partial_i\psi} \right) \delta\psi + \partial_i \left(\frac{\delta\mathscr{L}}{\delta\partial_i\psi} \delta\psi + \varepsilon^\alpha \delta^i_\alpha \mathscr{L} \right) \right\} \quad \textbf{(Eq. 14)}$$

If we let

$$\tau^i_{\alpha\beta} = -\frac{\delta\mathscr{L}}{\delta\partial_i\psi} f_{\alpha\beta}\psi \quad \textbf{(Eq. 15)}$$

$$\Sigma^i_\alpha = \delta^i_\alpha \mathscr{L} - \frac{\delta\mathscr{L}}{\delta\partial_i\psi} \partial_\alpha \psi \quad \textbf{(Eq. 16)}$$

$$\Sigma_{[ab]} = \partial_i \left(x_j \delta^j_{[a} \Sigma^i_{b]} \right) \quad \textbf{(Eq. 17)}$$

Application of the transformations to Eq. 14, yields the field equations

$$\partial_i \tau^i{}_{ab} - \Sigma_{[ab]} = 0, \qquad \partial_i \Sigma^i_a = 0 \quad \textbf{(Eq. 18)}$$

under a global Poincaré transformation. In a local Poincaré transformation,

$$\delta S = \int_\Omega d^4x \left\{ \left(-\partial_i \tilde{\omega}^{ab}\right)\left(\tau^i{}_{ba} + x_b \Sigma^i_a\right) + \left(\partial_i \tilde{\varepsilon}^a\right) \Sigma^i_a \right\}$$

$$= \int_\Omega d^4x \left\{ \left(-\partial_i \omega^{ab}\right) \tau^i{}_{ba} + \left(\partial_i \varepsilon^a - \omega^a_b \delta^b_i\right) \Sigma^i_a \right\} \quad \textbf{(Eq. 19)}$$

Local gauge invariance is preserved only if we introduce six rotational gauge potentials $\Gamma^{[\alpha\beta]}{}_i(x)$ and four translational gauge potentials $e_i{}^\alpha(x)$ so that $\mathscr{L}\left(\psi, \partial_i\psi\right) \rightarrow \mathscr{L}\left(\psi, \partial_i\psi, \Gamma^{[\alpha\beta]}_i, e^\alpha_i\right)$ and $\delta\Gamma^{ab}_i \sim \partial_i \omega^{ab}$, $\delta e^a_i \sim \omega^a_b \delta^b_i - \partial_i \varepsilon^a$, $\frac{\delta\mathscr{L}}{\delta\Gamma^{\alpha\beta}_i} \sim \tau^i_{\alpha\beta}$, $\frac{\delta\mathscr{L}}{\delta e^\alpha_i} \sim \Sigma^i_\alpha$. Now, under a global transformation, the rigidity condition $\pi\left(x^i \partial_i \psi\right) = \left(\pi x^a\right) \delta^i_a \partial_i \left(\pi\psi\right)$ holds. But under a local transformation.

$$\pi\left(\xi^i\partial_i\psi\right)=\left(\pi\xi^a\right)\delta^i_a\partial_i\left(\pi\psi\right)+\xi^i\left[-f_{ab}\partial_i\omega^{ab}+\left(\partial_i\varepsilon^a-\omega^a_b\delta^b_i\right)\partial_a\right]\psi$$
$$\approx\left(\pi\xi^a\right)\left(\delta^i_a+\delta e^i_a\right)\left(\partial_i+\delta\Gamma^{ba}_i f_{ab}\right)\left(\pi\psi\right) \qquad \textbf{(Eq. 20)}$$

This suggests the following minimal substitution schemes: $\delta^a_i\to e^a_i$, $\partial_i\to D_i=\partial_i+\Gamma^{ab}_i f_{ab}$. With these, the field transformation becomes $\psi(x)\to\left[\tilde\pi\psi\right](x)=\left(1+\omega^{ab}f_{ba}-\varepsilon^a D_a\right)\psi(x)$ and the Poincaré group algebraic relations becomes

$$\left[f_{\alpha\beta},D_\gamma\right]=g_{\gamma[\alpha}D_{\beta]} \qquad \textbf{(Eq. 21)}$$

$$\left[D_\alpha,D_\beta\right]=e^i_\alpha e^j_\beta\left(F^{\gamma\delta}_{ij}f_{\delta\gamma}-F^\gamma_{ij}D_\gamma\right) \qquad \textbf{(Eq. 22)}$$

where

$$F^{\gamma\delta}_{ij}:=2\left(\partial_{[i}\Gamma^{\gamma\delta}_{j]}+\Gamma^{\alpha\gamma}_{[i}\Gamma^{\beta\delta}_{j]}g_{\alpha\beta}\right)=-F^{\delta\gamma}_{ij} \qquad \textbf{(Eq. 23)}$$

$$F^\gamma_{ij}:=2\left(\partial_{[i}e^\gamma_{j]}+\Gamma^{\alpha\gamma}_{[i}e^\beta_{j]}g_{\alpha\beta}\right)=2D_{[i}e^\gamma_{j]} \qquad \textbf{(Eq. 24)}$$

The rigidity condition becomes $\tilde\pi\left(\xi^i\partial_i\psi\right)=\left(\tilde\pi\xi^\alpha\right)\left(e^i_\alpha+\delta e^i_\alpha\right)\left(D_i+\delta\Gamma^{\beta\alpha}_i f_{\alpha\beta}\right)\left(\tilde\pi\psi\right)$, giving us

$$\delta\Gamma^{\alpha\beta}_i=-D_i\omega^{\alpha\beta}-\varepsilon^\gamma F^{\alpha\beta}_{\gamma i} \qquad \textbf{(Eq. 25)}$$

$$\delta e^\alpha_i=\omega^\alpha_\beta e^\beta_i-D_i\varepsilon^\alpha-\varepsilon^\gamma F^\alpha_{\gamma i} \qquad \textbf{(Eq. 26)}$$

The variation of the Lagrangian gives

$$e\tau^i_{\alpha\beta}:=\frac{\delta\mathcal{L}}{\delta\Gamma^{\alpha\beta}_i}\equiv-\frac{\delta\mathcal{L}}{\delta\partial_i\psi}f_{\alpha\beta}\psi \qquad \textbf{(Eq. 27)}$$

$$e\Sigma^i_\alpha:=\frac{\delta\mathcal{L}}{\delta e^\alpha_i}=e^i_\alpha\mathcal{L}-\frac{\delta\mathcal{L}}{\delta\partial_i\psi}D_\alpha\psi \qquad \textbf{(Eq. 28)}$$

$$D_i e\tau^i{}_{\alpha\beta}-\Sigma_{[\alpha\beta]}=0 \qquad \textbf{(Eq. 29)}$$

$$D_i e\Sigma^i_\alpha=F^{\beta\gamma}_{\alpha i}e\tau^i{}_{\beta\gamma}+F^\beta_{\alpha i}\Sigma^i_\beta \qquad \textbf{(Eq. 30)}$$

where $e=\det e^i_j$. The field tensors given by Eqs. 23 and 24 may be compared with the Riemann tensor

$$R_{ijk}{}^m=e^\alpha_k e^m_\beta F_{ij\alpha}{}^\beta \qquad \textbf{(Eq. 31)}$$

and the Cartan torsion tensor,

$$S_{ijk}=\tfrac{1}{2}e^k_\alpha F_{ij}{}^\alpha \qquad \textbf{(Eq. 32)}$$

respectively. The simplest free field Lagrangian of the gauge fields is

$$\mathcal{L}_f\left(\Gamma,\partial\Gamma,e\right)=\frac{e}{2k}e^i_\alpha e^j_\beta F_{ji}{}^{\alpha\beta} \qquad \textbf{(Eq. 33)}$$

Minimization of the total action $S=\int\left(\mathcal{L}+\mathcal{L}_f\right)d^4x$ then gives

$$D_j\left(e e^i_{[\alpha}e^j_{\beta]}\right)=ek\tau^i{}_{\alpha\beta} \qquad \textbf{(Eq. 34)}$$

$$F_{ba}^{\ \ ib} - \tfrac{1}{2} e^{j}_{a} F_{bg}^{\ \ gb} = kS^{i}_{a} \qquad \textbf{(Eq. 35)}$$

For

$$T_{ij}^{\ \ k} := S_{ij}^{\ \ k} + 2d^{k}_{[i} S_{j]m}^{\ \ \ m} \qquad \textbf{(Eq. 36)}$$

$$G^{ij} := R^{ij} - \tfrac{1}{2} R g^{ij} \qquad \textbf{(Eq. 37)}$$

Eqs. (34) and (35) give

$$T^{ijk} = kS^{ijk} \qquad \textbf{(Eq. 38)}$$

$$G^{ij} = kS^{ij} \qquad \textbf{(Eq. 39)}$$

The former gives the torsion of the spacetime and eqn. latter is none other than Einstein's field equation.

4. PHYSICAL INTERPRETATION

In doing a Poincaré gauge transformation, we have shown that without any reference to geometry, we are able to obtain the Einstein's equation, albeit in a non-Torsion-free universe. Invariance under a local Poincaré transformation require the introduction of an antisymmetric rotational gauge potential, which is associated with spin. A translational gauge potential is likewise required and this is related to the energy-momentum tensor. A comparison of this theory and the geometric theory indicate that rotational potentials related to the connection, while translation potentials to the orientation of the local (vierbein) frames in spacetime. As hoped for, the dynamically defined currents (27) and (28) for local transformation agree with that of special relativity (15) and (16), for global transformation. The conservation law of energy-momentum (30) implies that both gauge fields act upon the corresponding source, while the conservation law of angular momentum (29) indicates that the gauge fields do not exert torques on the matter distribution. Equation (38) indicate that torsion exists only inside spinning matter. Hence spin manifests itself only by means of its influence on the metric tensor. A gauge theory of Poincaré transformation yields a non-torsion-free theory. In the macroscopic limit where spin averages out, the theory can reduce to Einstein's gravitational theory.

5. REFERENCES

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