Is the key to all knowledge, patience?

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Abstract: Viewing language as a collection of symbols, the written word may be taken as a finite sequence obtained from a finite collection of symbols. In this case, the answer to any question may be expressed as such a sequence. It is shown that, even in this context, it is not possible to exhaustively enumerate all finite sequences and find the answer to all questions. However, all such sequences may be systematically generated, thus guaranteeing finding a solution.

Key Words: finite sequences; cardinality; countability

1. INTRODUCTION

Stripped to its core, language is just a collection of symbols. They may be visual, oral, tactile, or any combination of these things among others. In this light, a problem or question may be viewed as a string of symbols. Necessarily, its solution, is also a string of symbols.

One ordinarily deals with a finite collection of symbols like the English language and its alphabet. Given this finite context, it is not unreasonable to wonder whether the brute force method of simply enumerating all possible finite sequence of symbols would generate all the answers to one's various questions.

We phrase this problem in a mathematical context and determine whether it is indeed a viable option.

2. FRAMEWORK

We do not address here questions of epistemology but rather take the simplistic view that well-posed questions are answerable. We only want to address the potential to procedurally generate the answers to meaningful problems.

We consider a language we will call $\mathcal{L}$ with the alphabet $\mathcal{A}$ having μ elements. A word is a finite string of elements taken from $\mathcal{A}$, and we denote the collection of all words by $\mathcal{W}$. We introduce the collection $\mathcal{W}$ in order to help in transitioning from ordinary language to the formal language $\mathcal{L}$. It will be apparent shortly that the use of $\mathcal{W}$ may be eliminated. Finally, we let $\mathcal{C}$ be the set of all compositions formed by a finite sequence of words from $\mathcal{W}$.

Let $\mathcal{A} = \{x_i : 1 = 1, 2, \ldots, \mu\}$. If $w \in \mathcal{W}$, then w is of the form $w = x_{i_1} x_{i_2} \cdots x_{i_k}$. If c is a composition from $\mathcal{C}$, then we may write $c = w_1 w_2 \cdots w_j$ for some $j \in \mathbb{Z}$. Indeed, one sees that since each w_j is a sequence of x_is, we may think of each c as a sequence of x_is. We will take this view of each composition from here on.

We further assume that there is a unique answer to each question. Formally, this assumption allows us to think of finding the answer to a question as evaluating a function f from $\mathcal{C}$ into $\mathcal{C}$. In theory, determining whether we could enumerate all possible answers to all possible questions may now be localized to analyzing the range of f. However, not knowing the answer to a question limits the usefulness of f. Indeed, our analysis must take into account all of $\mathcal{C}$. Determining whether we could exhaust all finite sequences from $\mathcal{A}$ translates to whether there are a finite number of all such sequences.

There is one final assumption that completes our framework. We must have some means of recognizing the correct answer. I think this is a deep philosophical question and we will avoid this. We simply assume that validity of a solution is instantly recognized in this codified language $\mathcal{L}$. This is another reason why it is handy to assume that each question has a unique answer.

3. THE NATURE OF $\mathcal{C}$

3.1 Finite sequences

Could the set of finite sequence from a finite collection of elements be exhaustively enumerated?

Let us consider first the finite sequences of length M. In this case, we may apply the multiplicative rule for counting to get

Proposition 1. Let S be the collection of all finite sequences, of length M, from a set $\mathcal{A}$ with μ elements. Then S has exactly μ^M elements. ■

Hence, all such sequences may be listed down in a finite amount of time, albeit finite in this case does not necessarily mean short.

If we know that the answer to our question may be given by a finite sequence of a known fixed length, Proposition 1 tells us that, indeed, finding our answer is only a question of patience. The problem of course is that not knowing the answer, we are unable to identify how long it will be. At best, we only concede that it must be expressible as a finite sequence.

In terms of the function f, the answer we are looking for is in the collection of all finite sequences with lengths l, $l = 1, 2, \ldots, n, \ldots$. And thus, we have

Proposition 2. The set $\mathcal{C}$ is infinite; that is, the number of all finite sequences from $\mathcal{A}$ is infinite.

Proof: $\mathcal{C}$ may be broken up into the collection of all finite sequences of length l as l ranges over all counting numbers. For each l, Proposition 1 tells us that we can find μ^l sequences.

Thus the set $\mathcal{C}$ has cardinality

$$|\mathcal{C}| = \sum_{l=1}^{\infty} \mu^l. \tag{1}$$

This gives us a geometric series which converges only if $|\mu| < 1$. In this case, since μ denotes the cardinality of the alphabet $\mathcal{A}$, $|\mu| \geq 1$. Hence, the series (1) diverges to infinity. ■

Proposition 2 tells us that it is impossible to find the answers to all questions by exhaustively enumerating all possible answers. The question remains whether one could even systematically enumerate all answers.

The question of whether one could enumerate the elements of a particular set reduces to a question of the nature of the infinity obtained. At present, we are certain that there are two types of infinities - the cardinality of the rational numbers and the cardinality of the real numbers.

3.2 Countability

The cardinality of a set is defined by a bijection between the set and a previously categorized set. When we talk of cardinality, we usually take the set of natural numbers, $\mathbb{N}$, to be the reference set. A set is said to be countable if it is either finite or may be placed in a one-to-one correspondence with $\mathbb{N}$.

We show that the $\mathcal{C}$ is countable. Since we have already shown that $\mathcal{C}$ is not finite, we will now have to show that we could exhibit a bijection between the $\mathcal{C}$ and $\mathbb{N}$.

First, we show that we could define a one-to-one function from the $\mathcal{C}$ into the set of natural numbers.

Proposition 3. There exists a one-to-one function from $\mathcal{C}$ into $\mathbb{N}$.

Proof: Recall that $\mathcal{A} = \{x_i : 1 = 1, 2, \dots, \mu\}$. We can define a bijection f from $\mathcal{A}$ onto $\{1, 2, \dots, \mu\}$ by taking $f(x_i) = i$.

Next, let $\{2, 3, 5, \dots, p_i, \dots\}$ be an enumeration of all the prime numbers. Now, let

$$c = a_1 a_2 \cdots a_l$$

be a composition in $\mathcal{C}$. Define the mapping ϕ from $\mathcal{C}$ into $\mathbb{N}$ by

$$\phi(c) = 2^{f(a_1)} 3^{f(a_2)} \cdots p_l^{f(a_l)}.$$

By the Fundamental Theorem of Arithmetic, ϕ is a one-to-one function which maps each composition c to a unique natural number. ■

Proposition 4. $\mathcal{C}$ is countable.

Proof: By Proposition 3, ϕ is a one-to-one map from $\mathcal{C}$ into $\mathbb{N}$. So the range of ϕ, call it E, is a subset of $\mathbb{N}$. Since every subset of a countable set is countable, E is countable.

Let g be a bijection from E onto $\mathbb{N}$. Observe that ϕ is a bijection from $\mathcal{C}$ onto E. Hence $g \circ \phi$ is a bijection from $\mathcal{C}$ onto $\mathbb{N}$. Consequently, $\mathcal{C}$ is countable. ■

3. SUMMARY AND CONCLUSIONS

In a sense, this paper addresses why there are unproven conjectures or why there are unsolved problems. The perspective taken is that a coherent thought may be expressed as a finite sequence of symbols taken from a finite alphabet, $\mathcal{A}$.

We pose the context that all questions and their answers, which we call $\mathcal{C}$, may be viewed as finite sequences from $\mathcal{A}$. Even in this context, it turns out that it is impossible to exhaustively enumerate all elements of $\mathcal{C}$, thus eliminating the brute force approach to finding answers to unsolved problems.

It was shown, however, that $\mathcal{C}$ while infinite, is countable. Being countable, a concrete procedure exists which allows the systematic generation of all its elements. Thus, given enough time, one may produce the finite sequence which answers one particular question.

4. REFERENCES

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